Describing Function

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Optimal Quasi-linearization

• Given an input \( u(.) \), optimally approximate the output of a nonlinear element \( N(.) \) by the output of linear time-invariant system \( h(.) \).
• Minimize the error criterion

\[
e^2(w) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{0}^{T} [y_N(t) - y_h(t)]^2 dt
\]

Most important case

\[ u(t) = u_0 \sin(\omega_0 t) \]

Sufficient Conditions

• For the limit to exist, the nonlinear element must be input-output stable (bounded output \( y_N(t) \) for any bounded input \( u(t) \))
• A linear element is BIBO stable if and only if its impulse response \( h(t) \) satisfies

\[
\int_{0}^{\infty} |h(t)| dt < \infty
\]

Fourier Series

• Sinusoids

\[
y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \right]
\]

\[
a_0 = \frac{1}{\pi} \int_{0}^{2\pi} y(t) dt, \quad a_n = \frac{1}{\pi} \int_{0}^{2\pi} y(t) \cos(n \omega_0 t) dt
\]

\[
b_n = \frac{1}{\pi} \int_{0}^{2\pi} y(t) \sin(n \omega_0 t) dt
\]

• Exponential form

\[
y(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn \omega_0 t}, \quad \alpha_n = \frac{1}{2\pi} \int_{0}^{2\pi} y(t)e^{-jn \omega_0 t} d\omega_0 t
\]
Theorem: Harmonic Balance
If the input to a nonlinear system $N(\cdot)$ is the sinusoid $u(t) = u_0 \sin(\omega_0 t)$ and its output has the first harmonic
$$y(t) = a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)$$
Then a linear system with proper rational transfer function $H(s)$ with poles in the LHP is an optimal quasi-linearization of $N(\cdot)$ w.r.t. $u(t)$ iff
$$H(j\omega_0) = \frac{b_1 + ja_1}{u_0}$$

Describing Function
- The describing function of a nonlinearity $N(\cdot)$ with input $u(t) = u_0 \sin(\omega_0 t)$ is the complex-valued function
$$N(u_0, j\omega_0) = N_r + jN_i = \frac{b_1 + ja_1}{u_0}$$
$$j2\alpha_1 = \frac{j}{\pi} \int_{0}^{2\pi} y(t)e^{-j\omega_0 t} d\omega_0 t$$
$$= \frac{j}{\pi} \int_{0}^{2\pi} y(t)\{\cos(\omega_0 t) - j \sin(\omega_0 t)\} d\omega_0 t$$
$$= b_1 + ja_1$$

Symmetry: Memoryless Nonlinearity
$$N(u), u(t) = u_0 \sin(\omega_0 t)$$
- Odd nonlinearity (odd input):
  $$y(t) = N(u(t)) = -N(-u(t))$$
  $$= -N(u(-t)) = -y(-t)$$
- Odd output has only odd terms in its Fourier series ($a_n = 0, n = 0,1, ...$)
- Real $N(u_0) = N_r$

Nonlinearity with memory
$$N(u, \dot{u})$$
- e.g. hysteresis, backlash
- The output depends on the sign of $\dot{u}$
- Cannot use symmetry properties
- Complex $N(u_0, j\omega_0) = N_r + jN_i$
Filtering Hypothesis

- Assume that the linear subsystem is sufficiently low-pass so that all the but the fundamental components are negligible.

\[
e(t) \xrightarrow{N(e)} G(s) \xrightarrow{y(t)}
\]

Example: Relay With Hysteresis

\[u(t) = A \sin(\omega_0 t), \quad d = A \sin(\theta_1)
\]

\[\theta_1 = \sin^{-1}(d/A) < \pi/2
\]

(for \(A < d\), output is fixed at \(\pm M\))

\[M \quad y(t) \quad \theta_1 \quad \theta_1 + \pi
\]

Describing Function Evaluation

\[
N(A, \omega) = \frac{j}{\pi A} \int_0^{2\pi} u(t)e^{-j\theta} d\theta
\]

\[= \frac{j}{\pi A} \left\{ \int_0^{\theta_1} (-M)e^{-j\theta} d\theta + \int_{\theta_1}^{\theta_1+\pi} Me^{-j\theta} d\theta + \int_{\theta_1+\pi}^{2\pi} (-M)e^{-j\theta} d\theta \right\}
\]

\[= \frac{M}{\pi A} \left\{ e^{-j\theta_1} - 1 + e^{-j\theta_1} - e^{-j(\pi+\theta_1)} + 1 - e^{-j(\pi+\theta_1)} \right\}
\]

Describing Function

\[
N(A, \omega) = \frac{4M}{\pi A} e^{-j\theta_1}
\]

\[= \frac{4M}{\pi A} \left\{ \cos(\theta_1) - j \sin(\theta_1) \right\}
\]

\[= \frac{4M}{\pi A} \left\{ \sqrt{1 - \left(\frac{d}{A}\right)^2} - j \left(\frac{d}{A}\right) \right\}
\]

\[|N(A, \omega)| = \frac{4M}{\pi A}
\]
Extended Nyquist Criterion

\[ e = -y = -G(j\omega)N(A, \omega)e \]

Characteristic equation as in Nyquist criterion
\[ 1 + G(j\omega)N(A, \omega) = 0 \]

- Treat \( N(A, \omega) \) as a complex gain and use the Nyquist criterion with \((-1,0)\) replaced by \(-1/N(A, \omega_0)\).

\[ e(t) \rightarrow N(e) \rightarrow G(s) \rightarrow y(t) \]

Extended Nyquist Criterion

Characteristic equation as in Nyquist criterion
\[ 1 + G(j\omega)N(A, \omega) = 0 \]

- For an open-loop stable \( G(j\omega) \), the system is closed-loop stable if its polar plot does not intersect a plot of \(-1/N(A, \omega_0)\).
- Intersections indicate a (possible) limit cycle.

Limit Cycle

Intersection
\[ G(j\omega_0) = -1/N(A, \omega_0) \] or equivalently
(i) \[ |N(A, \omega_0)| = 1/|G(j\omega_0)| \]
(ii) \[ \theta_G + \theta_N = (2m + 1)\pi, m = 0,1,\ldots \]

Solve for the intersection graphically or analytically to obtain
\( (A, \omega_0) = \text{(Amplitude, frequency)} \) of limit cycle.

Memoryless Nonlinearity

- \( N(A) \) real and the intersection can only occur on the real axis when

\[ \text{Im}\{G(j\omega_0)\} = 0 \]
\[ G(j\omega_0) = -1/N(A) \]

\( (A, \omega_0) = \text{(Amplitude, frequency)} \) of limit cycle.

Solve the first equation for \( \omega_0 \), then solve the second for \( A \).
Stability of the Limit Cycle

• **Stable**: points near the intersection on the increasing $A$ side of $1/N(A, \omega)$ not encircled by the polar plot of $G(j\omega)$, otherwise unstable.

$$-\frac{1}{N(A, \omega)}$$

Limit Cycle: Relay with Hysteresis

$$N(A, \omega) = \frac{4M}{\pi A} \left\{ \sqrt{1 - \left( \frac{d}{A} \right)^2} - j \left( \frac{d}{A} \right) \right\}$$

$$-\frac{1}{N(A, \omega)} = -\frac{\pi A}{4M} \left\{ \sqrt{1 - \left( \frac{d}{A} \right)^2} + j \left( \frac{d}{A} \right) \right\}$$

$$= -\frac{\pi}{4M} \left\{ \sqrt{A^2 - d^2} + jd \right\}$$

Imaginary part is a constant independent of $A$

Linear Subsystem

$$G(s) = \frac{20}{s^2 + 2s + 10}$$

$$G(j\omega) = \frac{20}{10 - \omega^2 + j2\omega}$$

$$= 20 \frac{10 - \omega^2 - j2\omega}{(10 - \omega^2)^2 + 4\omega^2}$$

$$Re\{G(j\omega)\} = \frac{200 - 20\omega^2}{\omega^4 - 16\omega^2 + 100}$$

$$Im\{G(j\omega)\} = \frac{-40\omega}{\omega^4 - 16\omega^2 + 100}$$

Numerical Values

• Let $M = 1, d = 0.2$

$$\frac{1}{N(A, \omega)} = -\frac{\pi}{4} \left\{ \sqrt{A^2 - 0.04} + j0.2 \right\}$$

$$Re\{G(j\omega)\} = \frac{200 - 20\omega^2}{\omega^4 - 16\omega^2 + 100}$$

$$= -\frac{\pi}{4} \sqrt{A^2 - 0.04}$$

$$Im\{G(j\omega)\} = \frac{-40\omega}{\omega^4 - 16\omega^2 + 100} = -\frac{0.2\pi}{4}$$
Analytical Solution

\[ \omega^4 - 16\omega^2 + \frac{800}{\pi} \omega + 100 = 0 \]

\( \omega_0 = 0.7072 \text{ rad/s} \)

\[ A = \sqrt{0.04 + \left[ \frac{1}{\pi} \times \frac{800 - 80\omega^2}{\omega^4 - 16\omega^2 + 100} \right]^2} \]

\( = 0.204 \)

Limit Cycle: Relay with Hysteresis

>> G=tf([20,1,2,10]);