OUTLINE

- Gain
- Small gain theorem
- Circle criterion

NORMS

- Norm of \( u(k), k = 0, 1, \ldots, \infty, u(k) \ m \times 1 \)
  \[ u = col\{u(k), k = 0, 1, \ldots, \infty\} \]
- Define the norm
  \[ \|u\|_s = \sqrt{\sum_{k=0}^{\infty} ||u(k)||^2} \]

GAIN

- Nonlinearity (assume zero at the origin)
  \[ y = N(u) \]
  \[ u = col\{u(k), k = 0, 1, \ldots, \infty\} \]
  \[ y(k) \ l \times 1, \quad u(k) \ m \times 1 \]
  \[ y = col\{y(k), k = 0, 1, \ldots, \infty\} \]
  \[ \gamma = \max_{\|u\|_s} \frac{\|y\|_s}{\|u\|_s} \]
**INPUT-OUTPUT STABILITY**

- For any bounded input sequence $\|u\|_s < K_u$, the output sequence is also bounded, i.e. $\|y\|_s < K_y$
- Input-output stable system must have an upper bound on the ratio of the norm of its output to that of its input.
- Upper bound = gain of nonlinear system
  
\[ \gamma = \max \frac{\|y\|_s}{\|u\|_s} \]

**EXAMPLE**

- Find the gain of the saturation nonlinearity

\[ y = N(u) = \begin{cases} 
  K u, & |u| < L \\
  K L, & u \geq L \\
  -K L, & u \leq -L 
\end{cases} \]

\[ K > 0, L > 0 \]

**SOLUTION**

The output is bounded by

\[ |y| = \begin{cases} 
  K |u|, & |u| < L \\
  K L, & |u| \geq L 
\end{cases} \]

The norm of the output is

\[ \|y\|_s^2 = \sum_{k=0}^{\infty} \|y(k)\|^2 \leq K \sum_{k=0}^{\infty} \|u(k)\|^2 \]

The gain of the system is the constant

\[ \gamma_{sat} = K \]

**EXAMPLE**

- Find the gain of a causal linear system with impulse response $G(k), k = 0,1,2,\ldots$

Solution

System response: input $u(k), k = 0,1,2,\ldots$

\[ y(k) = \sum_{i=0}^{\infty} G(i)u(k-i), k = 0,1,2,\ldots \]

\[ \|y\|_s^2 = \sum_{k=0}^{\infty} \|y(k)\|^2 = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{\infty} G(i)u(k-i) \right)^2 \]
**THEOREM: H∞ NORM**

\[ \gamma_G = \|G\|_s = \max_{\omega} \|G(e^{j\omega T})\| \]

**Proof**

Parseval’s Identity (stable system, finite energy for output)

\[ \|y\|_s^2 = \sum_{k=0}^{\infty} \|y(k)\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|y(e^{j\omega T})\|^2 d\omega \]

**EXAMPLE**

Verify that

\[ \gamma_G = \|G\|_s = \max_{\omega} \|G(e^{j\omega T})\| \]

can be used to find the gain of the linear system with transfer function

\[ G(z) = \frac{Z}{z - a}, \quad |a| < 1 \]
**SOLUTION**

Impulse response
\[ g(k) = a^k, \quad k = 0,1,2, \ldots, |a| < 1 \]
Gain \[ \|g\|_s = \sum_{k=0}^\infty |g(k)|^2 = \sum_{k=0}^\infty |a|^{2k} = \frac{1}{1-|a|^2} \]

Frequency response
\[ G(e^{j\omega T}) = \frac{e^{j\omega T}}{e^{j\omega T} - a} = \frac{1}{1 - ae^{-j\omega T}} = \frac{1}{1 - \cos(\omega T) + ja \sin(\omega T)} \]

Maximum magnitude (min denominator)
\[ \max_{\omega} \|G(e^{j\omega T})\| = \frac{1}{|1 - 2\cos(\omega T) + a^2|} = \frac{1}{1 - |a|^2} \]

**CLOSED-LOOP SYSTEM**

- Block diagram of a nonlinear closed-loop system.

**SMALL GAIN THEOREM**

The closed-loop system is input-output stable if the product of the gains of all the systems in the loop is less than unity, i.e.
\[ \prod_{i=1}^{2} \gamma_{N_i} < 1 \]

Can generalize to
\[ \prod_{i=1}^{n} \gamma_{N_i} < 1 \]

**PROOF**

Output \[ y = N_1(e) \]
Error \[ e = r - N_2(y) \]

Norm inequalities
\[ \|e\| \leq \|r\| + \gamma_{N_2} \|y\| \leq \|r\| + \gamma_{N_2} \gamma_{N_1} \|e\| \]

Solve for \( \|e\| \)
\[ \|e\| \leq \frac{\|r\|}{1 - \gamma_{N_2} \gamma_{N_1}} \]

which is finite if \( \prod_{i=1}^{n} \gamma_{N_i} < 1 \)
EXAMPLE

Simulate a feedback loop with linear subsystem transfer function ($T=0.01$ s)

$$G(z) = \frac{z + 0.5}{z^2 + 0.5z + 0.3}$$

in series with an amplifier of variable gain and a symmetric saturation nonlinearity with slope and saturation level unity. Investigate the stability of the system and discuss your results referring to the small gain theorem for two amplifier gains (a) 1, and (b) 0.2.

SOLUTION

Maximum magnitude of frequency response: obtained using the command

```
>> [mag, ph]=bode(g)
```

Maximum magnitude $\approx 5.126$.

$$\gamma_N < 1/5.126 \approx 0.195$$

INITIAL CONDITIONS

[1,0] AND UNITY GAIN.

Response of the closed-loop system with initial conditions [1,0] and unity gain (Unstable)

INITIAL CONDITIONS

[1,0] AND 0.2 GAIN

Response of the closed-loop system with initial conditions [1,0] and 0.2 gain (stable but small gain theorem fails).
**SECTOR BOUND NONLINEARITY**

A nonlinearity $N(.)$ belongs to the sector $[k_l, k_u]$ if it satisfies the condition

$$k_l \leq \frac{N(y)}{y} \leq k_u$$

$D(k_l, k_u)$ disk bounded by a circle with its center on the negative real axis with endpoints $-1/k_u$ and $-1/k_l$.

**ABSOLUTE STABILITY**

A nonlinear feedback system is absolutely stable with respect to the sector $[k_l, k_u]$ if the origin of the state space $x = 0$ is globally asymptotically stable for all nonlinearities belonging to the sector $[k_l, k_u]$.

**THE CIRCLE CRITERION**

Absolutely stable with stable $G(z)$ and nonlinearity $N(.)$ belonging to sector $[k_l, k_u]$ if

a) $0 < k_l < k_u$ and the Nyquist plot of $G(e^{j\omega T})$ does not enter or encircle the disc $D(k_l, k_u)$.

b) $0 = k_l < k_u$ & Nyquist plot is the right of $z = -1/k_u$

c) $k_l < 0 < k_u$ & Nyquist plot of $G(e^{j\omega T})$ inside $D(k_l, k_u)$

d) $k_l < k_u = 0$ & Nyquist plot of $G(e^{j\omega T})$ to the left of $z = -1/k_l$

e) $k_l < k_u < 0$ & Nyquist plot of $G(e^{j\omega T})$ does not enter or encircle the disc $D(k_l, k_u)$. 
**Case (a) $0 < k_l < k_u$**

- Diagram showing the region $D(k_l, k_u)$ in the $(y, k)$ plane.

**Case (b) $0 = k_l < k_u$**

- Diagram illustrating the boundary at $z = -1/k_u$.

**Case (c) $k_l < 0 < k_u$**

- Diagram depicting the region $D(k_l, k_u)$ with $k_l$ and $k_u$ labels.

**Case (d) $k_l < k_u = 0$**

- Diagram showing the region $D(k_l, k_u)$ with $k_u = 0$ and $z = -1/k_l$. 


EXAMPLE

For the furnace and actuator of Example 4-10, determine the stability sector for the system and compare it to the stable range for a linear gain block.

Solution

The transfer function of the actuator and furnace is

\[ G_a(z)G_{ZAS}(z) = 10^{-5} \frac{4.711z + 4.664}{z^3 - 2.875z^2 + 2.753z - 0.8781} \]

NYQUIST PLOT

- Plot lies inside a circle of radius \(10 + \epsilon\) which corresponds to a sector \((-0.1, 0.1)\).
- Plot is to the right of the line \(z = -0.935\) which corresponds to a sector \((0, 1.05)\).
- Cannot conclude that the system is stable in the sector \((-0.1, 1.05)\), i.e. cannot combine stability sectors.

JURY CRITERION

- Circle criterion gives conservative results: the actual stable sector may be wider than our estimates indicate.
- Jury criterion gives the stable range \((-0.099559, 3.47398)\). The lower bound is approximately the same as predicted by the circle criterion but the upper bound is more optimistic.
• Nonlinear system: saturation nonlinearity with a linear gain of unity and saturation levels (−1,1).
• System is stable outside range predicted by the circle criterion.