Lyapunov Stability Theory: Linear Systems

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Outline

- Lyapunov’s (first, indirect) linearization method.
- Linear time-invariant case.
- Domain of attraction.

Lyapunov’s Linearization Method

- Linearize nonlinear \( \dot{x} = f(x) \) system in vicinity of equilibrium \( x_e \):
  \[
  \Delta \dot{x} = \frac{\partial f(x)}{\partial x} \bigg|_{x_e} \Delta x.
  \]
- Find the eigenvalues of the linearized system.
  The equilibrium \( x_e \) of the nonlinear system is:
  - Exponentially stable if all the eigenvalues are in the open LHP.
  - Unstable if one or more of its eigenvalues is in the open RHP.
  - Inconclusive for LHP eigenvalues and one or more eigenvalues on the imaginary axis.

Example

- Determine the stability of the equilibrium of the mechanical system at the origin
  \[
  m\ddot{y} + b\dot{y} + k_1y + k_3y^3 = f
  \]
- Equilibrium with \( f = 0 \)
  \[
  \dot{y} = 0, \quad \ddot{y} = 0
  \]
  \[
  k_1y + k_3y^3 = 0
  \]
  \[
  y = 0
  \]
Nonlinear State Equations

- Physical state variables
  \[ x_1 = y, \quad x_2 = \dot{y} \]
- State Equations
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= \frac{1}{m} \left( f - bx_2 - k_1 x_1 - k_3 x_1^3 \right)
  \end{align*}
  \]
  \[
  \left. \frac{1}{m} \frac{\partial (f - bx_2 - k_1 x_1 - k_3 x_1^3)}{\partial x_1} \right|_{x_1=0} = -\frac{k_1}{m}
  \]

Linearization and Stability

- Equilibrium state \( x = [0 \ 0]^T \)
- Linearized model with \( m = 1 \)
  \[
  \Delta \dot{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -b \end{bmatrix} \Delta x
  \]
- Characteristic polynomial and stability
  \[
  \lambda^2 + b\lambda + k_1 = 0
  \]
  \[
  \lambda_{1,2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - k_1}
  \]
- Stable \( \text{Re}\{\lambda_{1,2}\} < 0 \)

Linear Time-invariant Case

The LTI system

\[ \dot{x} = Ax \]

is asymptotically stable if and only if for any positive definite matrix \( Q \) there exists a positive definite symmetric solution \( P \) to the Lyapunov equation

\[ A^T P + PA = -Q \]

Proof: Sufficiency

- Use a quadratic Lyapunov function
  \[ V(x) = x^T P x, \quad P > 0 \]
  \[
  \dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}
  \]
  \[
  = x^T A^T P x + x^T P A x
  \]
  \[
  = x^T [A^T P + PA] x = -x^T Q x
  \]
  \[
  A^T P + PA = -Q
  \]

\( V(x) > 0, \dot{V}(x) < 0 \) \( \Rightarrow \) globally exp. stable.
Proof: Necessity

- Let $Q > 0$, $A$ Hurwitz ($\text{Re}[\lambda_i(A)] < 0$)

$$P = \int_0^\infty e^{At}Qe^{At} \, dt$$

$$A^TP + PA = \int_0^\infty A^T e^{At}Qe^{At} \, dt$$

$$+ \int_0^\infty e^{At}Qe^{At}Adt$$

$$= \int_0^\infty \frac{d}{dt} \{e^{At}Qe^{At}\} \, dt = -Q$$

$$\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A, \lim_{t \to \infty} e^{At} = [0]$$

$P$ Symmetric Positive Definite

$$P^T = \int_0^\infty [(e^{At})^TQe^{At}]^T \, dt = P$$

$$x^TPx = \int_0^\infty x^Te^{At}Qs^TQ_s e^{At}x \, dt$$

$$= \int_0^\infty y(t)^Ty(t) \, dt$$

$y(t) = Q_s e^{At}x = 0, \forall t$ for some nonzero $x$

iff $(A, Q_s)$ is not an observable pair.

$P > 0$ for $(A, Q_s)$ observable.

Note: $Q$ can be positive semidefinite.

Uniqueness

$$A^TP + PA = -Q$$

$$A^TP_1 + P_1A = -Q$$

Subtract

$$A^T(P - P_1) + (P - P_1)A = [0]$$

$$e^{At} \{A^T(P - P_1) + (P - P_1)A\}e^{At} = [0]$$

$$= \frac{d}{dt} \{e^{At}(P - P_1)e^{At}\}$$

$$e^{At}(P - P_1)e^{At} \text{ constant if and only if}$$

$$P - P_1 = [0]$$

Remarks

- Recall that the original Lyapunov theorem only gives a sufficient condition.

- If we start with $P$ (i.e. with $V(x)$) and solve for $Q$, the condition the test may or may not work.

- If we start with $Q$ (i.e. with the derivative and we find a $P$ the condition is necessary and sufficient.
Example

Determine the stability of the system with state matrix

\[ A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \]

using the Lyapunov equation with \( Q = I_2 \).

Note: The system is clearly stable by inspection since \( A \) is in companion form.

Solution

\[ A^T P + PA = -Q, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \]

\[ \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \]

- Multiply

\[ \begin{bmatrix} -12p_{12} & -6p_{22} + p_{11} - 5p_{12} \\ -6p_{22} + p_{11} - 5p_{12} & 2p_{12} - 10p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \]

- Equate to obtain three equations in three unknowns.

Equivalent Linear System

\[ \begin{bmatrix} -12p_{12} & -6p_{22} + p_{11} - 5p_{12} \\ -6p_{22} + p_{11} - 5p_{12} & 2p_{12} - 10p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \]

Choose \( P = I \)

\[ A^T + A = -Q \]

\[ \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = -\begin{bmatrix} 0 & 5 \\ 5 & 10 \end{bmatrix} \]

- \( Q \) not positive definite.
- No conclusion: sufficient condition only.
- Choose \( Q \) and solve for \( P \).
MAPLE

Compute:
with(LinearAlgebra):
Transpose(A).P+P.A

Solve the equivalent linear system: M.p=-q
p is a vector whose entries are the entries of the P matrix, similarly define q
LinearSolve(M,B)

MATLAB

\[ A^T P + PA = -Q \]

- Solve a different equation.
- Identical to our equation with \( A \) replaced by \( A^T \).

\[ AP + PA^T = -Q \]

- Eigenvalues are the same!

MATLAB Example

```matlab
>> A=[0,1; -6,-5];
>> Q=eye(2);
>> P=lyap(A,eye(2));
>> eig(P)
ans =
  0.1098
  1.1236
```
To Get Earlier Answer

\[
P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix}
\]

\[
\gg \text{P} = \text{lyap}(A', \text{eye}(2))
\]

\[
P =
\begin{bmatrix}
1.1167 & 0.0833 \\
0.0833 & 0.1167 \\
\end{bmatrix}
\]

Domain (Ball, Region) of Attraction

- Region in which the trajectories of the system converge to an asymptotically stable equilibrium point.
- Difficult to estimate, in general.
- Can be estimated using the linearized system in the vicinity of the asymptotically stable equilibrium.

Example

\[
\begin{align*}
\dot{x}_1 &= 3x_2 \\
\dot{x}_2 &= -5x_1 + x_1^3 - 2x_2 \\
\end{align*}
\]
Equilibrium \( x_2 = 0, x_1(x_1^2 - 5) = 0 \)

\[
x_e = 0, (\pm \sqrt{5}, 0)
\]

Lyapunov function candidate for \( x_e = 0 \)

\[
V(x) = ax_1^2 - bx_1^4 + cx_1x_2 + dx_2^2
\]

\[
= \frac{c}{2}(x_1 + x_2)^2 + \left(a - \frac{c}{2} - bx_1^2\right)x_1^2 + \left(d - \frac{c}{2}\right)x_2^2
\]

Calculate \( \dot{V}(x) \)

\[
\dot{V}(x) = [2ax_1 - 4bx_1^3 + cx_2 - cx_1 + 2dx_2] \\
= \left[\begin{array}{c} 3x_2 \\
-5x_1 + x_1^3 - 2x_2 \\
\end{array}\right] \\
= (3c - 4d)x_2^2 + 2(d - 6b)x_1^3x_2 + 2(3a - 5d - c)x_1x_2 + cx_1^2(x_1^2 - 5)
\]

For \( d = 6b, c = 3a - 5d, b = 1, a = 12 \)

\[
\Rightarrow d = 6, c = 6
\]

\[
\dot{V} = -6x_2^2 + 6x_1^2(x_1^2 - 5) < 0, |x_1| < \sqrt{5}
\]

\[
V(x) = 3(x_1 + x_2)^2 + (9 - x_1^2)x_1^2 + 3x_2^2 > 0, |x_1| < 3
\]
Simulation Results

- The ball of attraction can be estimated to be $B = \{ x \in \mathbb{R}^2 : ||x|| < \sqrt{5} \}$
- Although for $D = \{ x \in \mathbb{R}^n : |x_1| < 1.6 \}$ we have $V(x) > 0, \dot{V}(x) < 0$, this region includes divergent trajectories because $D$ is not an invariant set. For example, the trajectory starting at $x_0 = [0,4]^T$ crosses $x_1 = \sqrt{5}$ then diverges.

Theorem 3.9

- Equilibrium $x_e$ of $\dot{x} = f(x)$, $V : D \rightarrow \mathbb{R}$, $f : D \rightarrow \mathbb{R}^n$

  I. $M \subset D$ compact set containing $x_e$, invariant w.r.t. the solutions of $\dot{x} = f(x)$
  
  II. $\dot{V}(x) < 0$, $\forall x \in M, x \neq x_e$
  $\dot{V}(x) = 0$, $x = x_e$

  Then $M \subset R_A$ the region of attraction of $x_e$

Proof

- Under the assumptions $E = \{ x \in M : \dot{V}(x) = 0 \} = x_e$
- $N = x_e$ is the largest invariant set in $E$
- By La Salle’s Theorem, every solution starting in $M$ approaches $N$ as $t \rightarrow \infty$, i.e. approaches $x$ as $t \rightarrow \infty$
- $M$ is an estimate of the domain of attraction.

Example

$\dot{x}_1 = 3x_2$
$\dot{x}_2 = -5x_1 + x_1^3 - 2x_2$
$\dot{V} = -6x_2^2 + 6x_1^2(x_1^2 - 5) < 0, |x_1| < \sqrt{5}$
$V(x) = 3(x_1 + x_2)^2 + (9 - x_1^2)x_1^2 + 3x_2^2 > 0, |x_1| < 3$

For $x_1 = \pm \sqrt{5}$
$V(x_2) = 6x_2^2 \pm 13.42x_2 + 35$
$\frac{dV(x_2)}{dx_2} = 12x_2 \pm 13.42 = 0, x_2 = \mp 1.1183$
Invariant Set

\[ V(x_2) = 6x_2^2 + 13.42x_2 + 35 \]

Minimum value at edge

\[ \frac{dV(x_2)}{dx_2} = 12x_2 + 13.42 = 0, \quad x_2 = \mp 1.1183 \]

\[ V(x) = 27.5, \quad x = [\sqrt{5}, -1.1183]^T \]

\[ x = [-\sqrt{5}, 1.1183]^T \]

\[ M = \{ x \in \mathbb{R}^2 : V(x) \leq 27.5 - \epsilon \} \]

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Estimate Using Linearized system

\[ \dot{x} = f(x), \quad x_e = 0 \]

\[ \dot{x} = \left[ \frac{\partial f(x)}{\partial x} \right]_0 x + g(x) = Ax + g(x) \]

\[ V(x) = x^T P x \]

Solve

\[ A^T P + PA = -Q \]

\[ \dot{V}(x) = x^T P x + x^T P \dot{x} \]

\[ = (Ax + g(x))^T P x + x^T P (Ax + g(x)) \]

\[ = -x^T Q x + 2g^T P x < 0 \]

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Example

\[ \dot{x} = Ax + g(x) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix} \]

Equilibrium \( x_e = 0 \)

Solve \( A^T P + PA = -2I_2 \Rightarrow P = I_2 \)

\[ V(x) = x^T x \]

\[ \dot{V}(x) = -x^T Q x + 2g^T P x \]

\[ = -2(x_1^2 + x_2^2) + 2x_2^3 \]

\[ = -2x_1^2 - 2x_2^2(1 - x_2) < 0 \]

for \( \|x\| < 1 \)

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Contours

[Diagram showing contours and vectors]