WHY STUDY NONLINEAR SYSTEMS?

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Outline

• Why study nonlinear systems?
• Classification of nonlinearities.
• Linear systems.
• Examples of nonlinear behavior.
• Equilibrium.

Why Nonlinear Control

1. Real-world systems are nonlinear time-varying.
2. Linearization only valid for a small operational range.
3. Some models cannot be linearized (discontinuous nonlinearities).
4. Robust designs can sometimes be obtained by introducing nonlinearity.
5. Nonlinear feedback can sometimes give simpler controllers.

Classification of Nonlinearities

I. Naturally occurring versus artificially introduced.
II. Continuous versus discontinuous.
I. Natural versus Artificial

**Natural**
All physical systems for example

**Mechanical:** Coulomb friction, stiction, square law friction.

**Electromagnetic:** hysteresis in $\phi$-i magnetization curves, relays, saturation.

**Electronics:** amplifier saturation.

**Artificial**
Introduced by control system designer

- Relay in home heating system.
- Thrusters in aerospace applications.
- Bang-bang control.
- Variable structure (switching) control.

II. Continuous vs. Discontinuous

**Continuous**
Linearization possible

$$f(x) - f(x_0) = \frac{df}{dx}\Delta x + O(\Delta x^2)$$

$$\Delta x = x - x_0$$

$$\frac{df}{dx} = \begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}$$

$$\Delta f \approx K\Delta x \quad \Delta x \text{ small}$$

Ex. $f = cv^2 \Rightarrow \Delta f = (2cv_0)\Delta v$

**Discontinuous (“hard”)**
Linearization is not possible.

- Hysteresis
- Dead zone.
- Backlash
- Stiction

Discontinuous (“hard”) Nonlinearities

![Amplifier saturation](image1)

- Motor dead zone
- Backlash in gears

Linearization

- For $\dot{x} = f(x, u, t), \dot{x}(t) = Ax(t) + Bu(t)$

$$A = \frac{\partial f}{\partial x}igg|_{(x_0, u_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1(x_0, u_0)} & \ldots & \frac{\partial f_1}{\partial x_n(x_0, u_0)} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1(x_0, u_0)} & \ldots & \frac{\partial f_m}{\partial x_n(x_0, u_0)} \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}igg|_{(x_0, u_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1(x_0, u_0)} & \ldots & \frac{\partial f_1}{\partial u_m(x_0, u_0)} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial u_1(x_0, u_0)} & \ldots & \frac{\partial f_m}{\partial u_m(x_0, u_0)} \end{bmatrix}$$
Linear Systems

State-space model
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]
- Obey the principle of superposition.
- Unique equilibrium point for \( A \) nonsingular.

LTI Systems:
- Asymptotic stability: LHP eigenvalues.
- Asymptotic stability \( \implies \) BIBO stability.
- Sinusoidal input \( \implies \) Sinusoidal output of same frequency.

Examples of Nonlinear Behavior
- Violates superposition.
- Multiple equilibrium points.
- Finite escape time.
- Limit cycles.
- Bifurcation.
- Chaos.

Finite Escape Time
\[ \dot{x} = -x + x^2, \quad x(0) = x_0 \]
- Solve by separation of variables
  \[ x(t) = \frac{x_0e^{-t}}{1 - x_0 + x_0e^{-t}} \]
- Equilibrium Points \( x_e = 0, 1 \)
- Solve
  \[ 1 - x_0 + x_0e^{-t} = 0 \]
- Finite escape time
  \[ t_e = \ln \left( \frac{x_0}{x_0 - 1} \right) \]
- \( x(t) \) goes to infinity in a finite time if its initial value is greater than unity.

Autonomous Systems
- Nonlinear \( \dot{x} = f(x, t, u), \quad y = h(x, t, u) \)
  \[
  f, \quad x \in \mathbb{R}, \quad h, \quad y = 1, \quad u \in \mathbb{R} \times 1
  \]
- Unforced systems \( u(t) = 0, \forall t \)
  \[ \dot{x} = f(x, t, 0) = \bar{f}(x, t) \]
- Forced system with known \( u(t), \forall t \), can be written in this form (equivalent to autonomous).
- **Autonomous System**: time-invariant and unforced
  \[ \dot{x} = f(x), y = h(x) \]
- Invariant with shifting the time origin.
Equilibrium Point $x_e$ of $\dot{x} = f(x)$

- System remains at $x_e$ if it is initially at $x_e$.
  \[ x(t_0) = x_e \Rightarrow x(t) = x_e, \forall t \geq t_0 \]
- Solve for equilibrium using $\dot{x} = 0$
- Consider 3 cases:
  - Linear time-invariant (LTI)
  - Nonlinear Autonomous
  - Time-varying

Linear Time-invariant (LTI)

- Model $\dot{x} = Ax$
- Equilibrium $Ax_e = 0$
- Nonsingular state matrix: unique solution $x_e = 0$
- Singular state matrix: infinitely many solutions (eigenvectors of zero eigenvalue)
- Response depends on eigenvalues of $A$
- Can determine stability by finding the eigenvalues (or the solution).

Nonlinear Autonomous Systems

- Model $\dot{x} = f(x)$
- Equilibrium $f(x_e) = 0$, multiple solutions possible but some may be inadmissible for physical reasons.
- Cannot solve for response except in special cases.
- Need indirect approach to qualitatively understand the system’s behavior.

Example

$$\dot{x} = -x + x^2, x(0) = x_0$$

$$-x + x^2 = x(x - 1) = 0$$

- Equilibrium Points $x_e = 0, 1$
- Alternatively, solve by separation of variables
  \[ x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}} \]

- Equilibrium Points $x_e = 0, 1$, since the system remains at either point if $x_0 = x_e$
Time-varying Systems

• Model
\[ \dot{x} = f(x, t) \]

• Equilibrium
\[ f(x_e(t_0), t_0) = 0, \forall t \geq t_0, \]
multiple solutions but the solution depends on \( t_0 \)

• We say \( x_e \) is an equilibrium point of the system at \( t_0 \).

• For linear time-varying systems where \( A(t) \) is nonsingular \( \forall t \geq t_0 \)

• More difficult, examined later.