Signal Detection

M. Sami Fadali
Professor of Electrical Engineering
University of Nevada, Reno

Outline
- Hypothesis testing.
- Neyman-Pearson (NP) detector for a known signal in white Gaussian noise (WGN).
- Matched filter implementation.
- Detection performance.

Detection Theory

Detection Problem: Decide based on noisy measurements with a measure of confidence
1. If an event occurred or not.
2. Which of a number of possible outcomes has occurred?
Statistics: called decision theory.

Example Applications of Detection Theory
- Radar/sonar.
- Communications.
- Speech/image processing.
- Biomedicine.
- Fault detection.
- Seismology.
Radar/Sonar

- Detect the presence of a target.
- Transmit an electromagnetic pulse.
- Test received noisy signal:
  - If a pulse is detected, it was reflected from the target and a target is present.
  - If no pulse is detected then no target is present.

Classes of Detection Problems

Detect signals from noisy measurements
1. Known signals in additive noise.
2. Deterministic signals with unknown parameters in additive noise.
3. Random signals in additive noise.
   - Typically, assume additive Gaussian noise.

Known Signals

- Using one or more noisy measurements decide if the known signal “s” is present.
- Additive noise: $x(n) = s(n) + w(n)$
- Binary Hypothesis: $H_0$ or $H_1$
- Statistical Decision: $D_0$ or $D_1$
- Probability density functions:
  $$ f(x|H_i), \quad i = 0,1 $$

Possible Outcomes of Binary Experiment

1. $D_0 \mid H_0$: correct decision.
2. $D_1 \mid H_0$: Type I Error, “false alarm”.
3. $D_1 \mid H_1$: correct decision.
4. $D_0 \mid H_1$: Type II Error, “miss”.
Radar/ Sonar Terminology

- Decompose observation space $\mathbb{R}$ into two disjoint subspaces: $\mathbb{R} = R_0 \oplus R_1$, $R_0 \cap R_1 = \emptyset$
- Define the following probabilities:
  1. False alarm
     $$P_{FA} = P(D_1|H_0) = \int_{R_1} f(x|H_0)dx$$
  2. Proper dismissal
     $$1 - P_{FA} = P(D_0|H_0) = \int_{R_0} f(x|H_0)dx$$

Probability of Detection

- $R_0 = \{x: x < \gamma\}$, $R_1 = \{x: x \geq \gamma\}$

Likelihood Ratio (LR)

- $$l(x) = \frac{f(x|H_1)}{f(x|H_0)} > \frac{D_1}{D_0}$$
- $\gamma$ = threshold
- Different detectors yield different thresholds.
- Use the likelihood ratio to decide.

Log-likelihood Ratio (LLR)

- Both sides of the inequality are positive.
- Log is an order preserving transformation.
- Often more convenient (exponential pdfs).
- $$L(x) = \ln[l(x)] = \ln\left[\frac{f(x|H_1)}{f(x|H_0)}\right] > \ln[\gamma]$$
Neyman-Pearson Criterion

- Select the threshold level $\gamma$
- NP Criterion: Maximize the probability of detection
  $$P_D = P(D_1|H_1) = \int_{R_1} f(x|H_1) \, dx$$
  such that the probability of false alarm $P_{FA}$ remains below a specified level $\alpha$.
- Maximize s.t. the constraint $P_{FA} = \alpha$

Matched Filters

- Known deterministic signal $s(n)$ in white Gaussian noise $w(n)$
- NP criterion: upper bound on probability of false alarm.
- Detection: distinguish between two hypotheses
  $H_0$: $x(n) = w(n)$, $n = 0, 1, \ldots, N - 1$
  $H_1$: $x(n) = w(n) + s(n)'$
  $w(n) \sim \mathcal{N}(0, \sigma^2)$
  $r_{ww}(k) = E\{w(n)w(n+k)\} = \sigma^2 \delta(k)$

Likelihood Ratio

$$l(x) = \frac{f(x|H_1)}{f(x|H_0)} > \gamma$$
$$D_1$$
$$D_0$$

$$x = [x(0), x(1), \ldots, x(N-1)]^T$$

$$f(x|H_1) = (2\pi\sigma^2)^{-N/2} \exp\left\{-\frac{(x-s)^T(x-s)}{2\sigma^2}\right\}$$

$$f(x|H_0) = (2\pi\sigma^2)^{-N/2} \exp\left\{-\frac{x^T x}{2\sigma^2}\right\}$$

Simplify Likelihood Ratio

$$l(x) = \frac{f(x|H_1)}{f(x|H_0)} = \frac{(2\pi\sigma^2)^{-N/2} \exp\left\{-\frac{(x-s)^T(x-s)}{2\sigma^2}\right\}}{(2\pi\sigma^2)^{-N/2} \exp\left\{-\frac{x^T x}{2\sigma^2}\right\}}$$

$$D_1$$
$$D_0$$

$$l(x) = \exp\left\{\frac{2s^T x - s^T s}{2\sigma^2}\right\} > \gamma$$

Log-likelihood Ratio

\[ L(x) = \ln[l(x)] = \frac{2s^T x - s^T s}{2\sigma^2} < \ln[y] \]
\[ T(x) = s^T x > \sigma^2 \ln[y] + \frac{1}{2} s^T s = \gamma' \]

NP Detection: Choose \( \gamma' \) to maximize \( P_D \) and satisfy the false alarm rate constraint

\[ P_{FA} = \int_{R_1} f(x|H_0)dx = \alpha \]

Replica Correlator

- \( T(x) = s^T x \) is the correlation of the data and the known signal (or its replica).
- \( T(x) \) weights the data samples with the values of the signal.
- The larger the signal amplitude the higher the weight.
- Negative amplitudes have negative weights.

Replica Correlator Block Diagram

\[ x(n) \]
\[ + \]
\[ \sum_{n=0}^{N-1} x(n)s(n) \]
\[ T(x) \]
\[ D_1 \]
\[ > \]
\[ \gamma' \]
\[ < \]
\[ D_0 \]

Matched Filter

- Match filter impulse response to the signal.
- Obtain impulse response by
  i. Flipping the signal.
  ii. Right shifting the signal by \( N - 1 \).

\[ h(n) = \begin{cases} s(N - 1 - n), & n = 0, 1, ..., N - 1 \\ 0, & \text{elsewhere} \end{cases} \]

\[ y(n) = \sum_{k=0}^{n} h(n - k)x(k) = \sum_{k=0}^{n} s(N - 1 - n + k)x(k), \quad n \geq 0 \]

\[ y(N - 1) = \sum_{k=0}^{N-1} s(k)x(k) = s^T x = s^T (s + w) \]
Matched Filter Block Diagram

\[ x(n) \rightarrow h(n) \rightarrow T(x) \]

\[ \text{Close at } n = N - 1 \]

\[ \gamma' \]

\[ D_1 > \]

\[ D_0 \]

Matched Filter Remarks

- Derivation requires known arrival time.
- Filter output is maximum at \( N - 1 \)
  \[ T(x) = \sum_{k=0}^{N-1} s(k)x(k) = s^T x = s^T (s + w) \]
  \[ w \sim N(0, \sigma^2 I_N) \]
- Signal energy \( \varepsilon = s^T s = \|s\|^2 \), the square of the norm of the signal vector.
- For no additive noise, \( T(x) = \) signal energy

Output of Filter: Mean & Variance

- For any filter with impulse response \( h \)
  \[ y(N - 1) = \sum_{k=0}^{N-1} h(N - 1 - k)x(k) \]
  \[ = h^T x = h^T (s + w) \]
  \[ E\{y(N - 1)|H_1\} = h^T (s + E\{w\}) = h^T s \]
  \[ \text{var}\{y(N - 1)|H_1\} = E\left\{ (h^T w)^2 \right\} \]
  \[ = h^T E\{ww^T\} h = \sigma^2 h^T h \]

Signal to Noise Ratio (SNR)

- SNR ratio of the signal \( \eta \): ratio of signal energy to noise energy.
  \[ \eta = \frac{E\{y(N - 1)|H_1\}^2}{\text{var}\{y(N - 1)|H_1\}} \]
  \[ = \frac{(h^T s)^2}{\sigma^2 h^T h} \]
  \[ = \frac{\left[ \sum_{k=0}^{N-1} h(N - 1 - k)s(k) \right]^2}{\sigma^2 \sum_{k=0}^{N-1} h^2(k)} \]
Matched Filter Maximizes SNR

- Consider all possible FIR filters.
  \[ \eta = \frac{\left( h^T s \right)^2}{\sigma^2 h^T h} \leq \frac{\| h \|^2 \| s \|^2}{\sigma^2 \| h \|^2} = \frac{\| s \|^2}{\sigma^2} = \frac{\mathcal{E}}{\sigma^2} \]
  \[ \mathcal{E} = \| s \|^2 = \sum_{k=0}^{N-1} s^2(k), \quad \eta_{\max} = \frac{\mathcal{E}}{\sigma^2} \]

- Maximum signal energy at the output of an FIR filter if \( h \) and \( s \) are collinear, \( h^T s = \mathcal{E} \).
- Performance of matched filter increases monotonically with the maximum SNR \( \eta \).

Matched Filter Performance

- Obtain \( P_D \) for a given \( P_{FA} \).
- \( T(x) \) Gaussian: linear combination of Gaussian
  a) Mean of \( T(x) = x^T s \) under \( H_0 \)
  \[ E\{ T(x) | H_0 \} = E\{ w^T s \} = E\{ w^T \} s = 0 \]
  b) Mean of \( T(x) = x^T s \) under \( H_1 \)
  \[ E\{ T(x) | H_1 \} = E\{ (s + w)^T s \} = s^T s = \mathcal{E} \]

Variance of \( T(x) = x^T s \)

- Applies under \( H_0 \) or \( H_1 \)
  \[ \text{var}\{ T(x) | H_i \} = E\{ (w^T s)^2 \} \]
  \[ = s^T E\{ w w^T \} s \]
  \[ = \sigma^2 s^T s \]

\[ \text{var}\{ T(x) | H_i \} = \sigma^2 \mathcal{E}, \quad i = 1, 2 \]

Distribution of \( T(x) \)

\[ T(x) \sim \begin{cases} \mathcal{N}(0, \sigma^2 \mathcal{E}), & H_0 \\ \mathcal{N}(\mathcal{E}, \sigma^2 \mathcal{E}), & H_1 \end{cases} \]

\[ T'(x) = T(x)/\sqrt{\sigma^2 \mathcal{E}} \quad \text{for } i = 1, 2 \]

\[ T'(x) \sim \begin{cases} \mathcal{N}(0,1), & H_0 \\ \mathcal{N}\left(\frac{\mathcal{E}}{\sqrt{\sigma^2}}, 1\right), & H_1 \end{cases} \]

- Performance improves as \( \mathcal{E}/\sigma^2 \) increases(SNR).
- Densities move farther apart but have the same shape.
**Threshold Calculation**

\[ P_{FA} = P(T > \gamma' \mid H_0) \]
\[ = P \left( T' > \frac{\gamma'}{\sqrt{\sigma^2 \epsilon}} \mid H_0 \right), \quad T'(x) = \frac{T(x)}{\sqrt{\sigma^2 \epsilon}} \]
\[ = Q \left( \frac{\gamma'}{\sqrt{\sigma^2 \epsilon}} \right) \]
\[ = \text{right tail probability for } \mathcal{N}(0,1) \]

- Solve for \( \gamma' \)
  \[ \gamma' = \sqrt{\sigma^2 \epsilon} \times Q^{-1}(P_{FA}) \]

**NP Detection Probability**

\[ P_D = P(T(x) > \gamma' \mid H_1), \quad T \sim \mathcal{N}(\epsilon, \sigma^2 \epsilon) \]
\[ \gamma' = \sqrt{\sigma^2 \epsilon} \times Q^{-1}(P_{FA}) \]

- Normalize: standard normal
  \[ P_D = Q \left( \frac{\gamma' - \epsilon}{\sqrt{\sigma^2 \epsilon}} \right) \]
  \[ = Q \left\{ \frac{\sqrt{\sigma^2 \epsilon} \times Q^{-1}(P_{FA}) - \epsilon}{\sqrt{\sigma^2 \epsilon}} \right\} \]
  \[ = Q \left\{ Q^{-1}(P_{FA}) - \sqrt{\epsilon/\sigma^2} \right\} \]

**Non-Gaussian Noise**

- Known deterministic signal in white Gaussian noise: NP criterion and the max SNR criterion give the matched filter detector.
- Non-Gaussian noise
  1. Matched filter detector is not NP optimal but still maximizes SNR.
  2. NP filter is nonlinear but the linear filter works well for moderate deviations from Gaussian.

**References**