Nyquist Stability Criterion

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Why use the Nyquist Criterion?
Answers the questions:
Q1. Are there any closed-loop poles in the RHP?
Q2. If the answer to Q1 is yes, then how many?

\[ T(s) = \frac{G(s)}{1 + G(s)H(s)} \]

Closed-loop Characteristic Equation

\[ 1 + G(s)H(s) = 1 + L(s) = 0 \]

\[ F(s) = 1 + L(s) = 1 + \frac{N_L(s)}{D_L(s)} = \frac{N_L(s) + D_L(s)}{D_L(s)} \]

- Zeros of \( F(s) \) are closed-loop poles
- Poles of \( F(s) \) are open-loop poles

\( F(s) \) Component Vectors

- For any value of the complex number \( s \), each component is a vector \( s + a \)
- Pole/zero at \( -a \)
Contour

- Contour = closed directed simple (does not cross itself) curve.

\[ \text{Contour Mapping} \]

\[ F(s) = \frac{(s + a)(s + b)}{(s + c)(s + d)} \]
\[ \angle F(s) = \angle(s + a) + \angle(s + b) - [\angle(s + c) + \angle(s + d)] \]

- Consider the angle change for \( F(s) \) as \( s \) traverses a known contour \( D \).
- Change in \( \angle F(s) \)
  \[ = \text{sum of angle changes for its zeros} - \text{sum of angle changes of its poles.} \]

Pole/Zero Location

- Zero net angle change for pole/zero outside the contour.
- \( \pm 1 \) encirclement = \( \pm 360^\circ \) net angle change for zero/pole inside the contour, respectively.

\[ \text{Angle Change for } F(s) = 1 + L(s) \]

Zeros/Poles of \( F(s) \) = closed-loop/open-loop poles
No. Encirclements of origin = (No. closed-loop poles inside contour) – (No. open-loop poles inside contour)
### Nyquist Contour
- Encloses all RHP poles and zeros of \( F(s) = 1 + L(s) \)
- Use net angle change to determine stability

### Mapping of \( L(s) \)
Assume real coefficients
- Plot \( L(j\omega) \) = polar plot.
- Plot \( L(-j\omega) \) = complex conjugate of polar plot
  = mirror image of polar plot
  \( L(s) \), large \(|s|\)
  \(|L(s)| \approx 0, \deg[N_L] < \deg[D_L]\)
  \(|L(s)| = \text{constant}, \deg[N_L] = \deg[D_L]\)

### Principle of the Argument
- If \( F(s) = 1 + L(s) \) has \( Z \) zeros and \( P \) poles inside the Nyquist contour (in RHP), a plot of \( F(s) \) as \( s \) travels once (clockwise) around the contour encircles the origin of the complex plane in which it is plotted \((-N)\) times where \((-N) = Z - P\)
  = no. of clockwise encirclements
i.e. \( N = P - Z\)
  = no. of counterclockwise encirclements

### Stability Results
(i) For stability (no closed-loop poles in the RHP)
  \( Z = 0 \) i.e. \( N = P \)
  For an open-loop stable system \((P = 0)\), \( N = 0 \)
(ii) For an unstable system \((Z \neq 0)\)
  \( Z = P - N\)
  = number of closed-loop poles in the RHP
(iii) Number of encirclements of the origin by \( F(s) = 1 + L(s) \)
  = number of encirclements of \((-1, 0)\) by \( L(s) \)
Nyquist Plot

Unity shift

\[ F(s) = 1 + L(s) \]

Nyquist Stability Criterion

**Necessary and sufficient** condition for the closed-loop stability of the loop gain \( L(s) \) is:

\[ N = P \]

\( N = \) no. of counterclockwise encirclements of \((-1,0)\) by \( L(s) \)

\( P = \) no. of RHP open-loop poles of \( L(s) \)

**Note:** Closed-loop stability is when \( Z = 0 \)

RHP Closed-loop Poles

1. **For an unstable system**, the number of closed-loop system poles in the RHP = \( Z = P - N \)
2. **For open-loop stable systems** (\( P = 0 \))
   (a) The system is stable with no \((-1,0)\) encirclements
   (b) Number of RHP poles for an unstable system = \(-N\) = number of clockwise encirclements of \((-1,0)\)

Open-loop Unstable System

\[ L(s) = \frac{K}{s - 2} \]

\( K = 1: N = 0 \quad Z = 1 \)

\( K = 4: N = 1 \quad Z = 0 \)

\( P = 1 \)
Root Locus: Open-loop Unstable System

$L(s) = \frac{K}{s - 2}$

Nyquist Plot: 3rd Order, Type 0

$L(s) = \frac{K}{(s + 1)(s/2 + 1)(s/3 + 1)}$

$P = 0$
$K = 2$: $N = 0$, $Z = 0$
$K = 12$: $N = -2$, $Z = 2$

Simplified Nyquist Criterion

- Assume that the loop gain function has no RHP open-loop poles (i.e. $P = 0$).
- Observer follows the polar plot of the loop gain in the direction of increasing frequency.
- The closed-loop system is stable if and only if the point $(-1,0)$ is to the left of the observer.
- Remark: The criterion implies no encirclements of $(-1,0)$ by the complete Nyquist plot.
Example: Unstable System

\[ L(s) = \frac{12}{(s + 1)(s/2 + 1)(s/3 + 1)} \]

Imaginary Poles

- Add a small semicircle around the origin for systems of type \( \geq 1 \)
- The small semicircle is mapped to \( l \) infinite semicircles traversed clockwise for a type \( l \) system.
- Similarly, add a small semicircle around any imaginary axis pole.

Modified Nyquist Contour

For small \( s = \epsilon e^{j\theta} \), \( |s| = \epsilon \)

\[ G(s) = K_e \frac{\prod_{k=1}^{m} \left( \frac{s}{\omega_k} + 1 \right)}{s^l \prod_{i=1}^{n-l} \left( \frac{s}{\omega_i} + 1 \right)} \approx \frac{K_e}{s^l} \]

- Net rotation for \( s = +\pi \)
- Denominator angle = \( l \pi \)
- Transfer function angle = \(-l \pi \) (\( l \) clockwise half circles)
Nyquist Plot: Type I, $P = 0$

Type I: one clockwise half circle

$P = 0$ (no RHP open-loop poles)

$N = 0$  $Z = 0$

Simplified Nyquist: Type I

$N(s) = \frac{K}{s(s/2 + 1)}$

$P = 0$

$N = 0$

Stable

Nyquist Plot: Type II, $P = 0$

Type II: two clockwise half circles

$P = 0$

$N = 2$  $Z = 2$

Simplified Nyquist: Type II

$N(s) = \frac{K}{s^2(s/2 + 1)}$

$N \neq 0$

Unstable
Nyquist Plot: Variable $K$

- Closed-loop characteristic equation
  \[ 1 + KG(s)H(s) = 1 + KL(s) = 0 \]
- Divide equation by $K$
  \[ 1/K + L(s) = 0 \]
- Repeating earlier analysis gives the same results with the point $(-1,0)$ replace by the point $(-1/K, 0)$
- Point moves as $K$ varies.

Nyquist Plot: Variable $K$

- Plot frequency response for a gain $K = 1$.
- Count encirclements of the point $-1/K$. 

\[-1/K_{cr} = -1/60\]