Phase Margin & Gain Margin

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Stability Margins

• What is the worst perturbation of the transfer function that will make the system marginally stable?
• Marginal stability for open loop stable systems is when the contour goes through the point \((-1,0)\).

Gain and Phase Margins

Gain Margin: gain perturbation that makes the system marginally stable.
Phase Margin: negative phase perturbation that makes the system marginally stable.

Model Perturbation

\[ \Delta G(s) = \text{model perturbation} \]
Gain Margin: \( \Delta G(s) = \Delta K \) (gain perturbation)
Phase Margin: \( \Delta G(s) = e^{-j\Delta \theta} \) (phase lag perturbation)
Definitions of Margins

Gain Margin: additional gain that makes the system on the verge of instability.
Phase Margin: additional phase lag that makes the system on the verge of instability.

Margins on Polar Plot

\[ \frac{\omega_{gc}}{\omega_{pc}} = \text{gain crossover (} M = 1 \text{)} \text{/ phase crossover (} \phi = -180^\circ \text{)} \]

\[ G(s) = \frac{10}{s\left(\frac{s}{5} + 1\right)\left(\frac{s}{20} + 1\right)} \]

Polar plot of Unstable System

\((-1,0)\) Point on Bode Plot

- Negative real axis for Nyquist plot corresponds to an angle of \(-180^\circ\).
- Magnitude of unity corresponds to zero dB.
- For PM we need to find the phase angle at magnitude unity (0dB).
- For GM we need to find the magnitude at an angle of \(-180^\circ\).

Negative GM (dBs) and PM.
Bode Plot of Stable System

MATLAB Margin

>> [Gm,Pm,Wcg,Wcp] = margin(g)

Gm = 2.5000
Pm = 22.5359
Wcg = 10.0000 (phase crossover freq.)
Wcp = 6.0783 (gain crossover freq.)

Gain Margin Calculation

• Multiply numerator and denominator by the complex conjugate of the denominator.

\[ G(j\omega) = \frac{N(j\omega)}{D(j\omega)} \times \frac{D^*(j\omega)}{D^*(j\omega)} = \frac{N(j\omega)D^*(j\omega)}{|D(j\omega)|^2} \]

• Equate the imaginary part of the numerator to zero and solve for the phase crossover frequency:

\[ Im[N(j\omega_{pc})D^*(j\omega_{pc})] = 0 \]

• Calculate the gain margin \( GM = -1/G(j\omega_{pc}) \)

Example: Gain Margin

• Solve for phase crossover (imaginary part zero)

\[ G(j\omega) = \frac{1000}{j\omega(j\omega + 5)(j\omega + 20)} \]

\[ G(j\omega) = \frac{-j(-j\omega + 5)(-j\omega + 20)}{1000(\omega^2 + 25)(\omega^2 + 400)} \]

\[ Im\{G(j\omega)\} = 0 \]

\[ \Leftrightarrow Re\{(−j\omega + 5)(−j\omega + 20)\} = 0 \]

\[ \Leftrightarrow 100 - \omega^2 = 0 \Leftrightarrow \omega_{pc} = 10 \text{ rad/s} \]
Calculate Gain Margin

• Evaluate the magnitude at the phase crossover frequency

\[ GM = -\frac{1}{G(j10)} \]
\[ = -\frac{(j10)(j10 + 5)(j10 + 20)}{1000} = 2.5 \]

Phase Margin Calculation

• Solve for gain crossover frequency (unity magnitude)

\[ |G(j\omega_c)|^2 = 1 \iff |N(j\omega_c)|^2 = |D(j\omega_c)|^2 \]

• Calculate the phase margin

\[ PM = 180^\circ + \angle G(j\omega_c) \]

Phase Margin Calculation

• Solve for gain crossover (unity magnitude)

\[ G(j\omega) = \frac{1000}{j\omega(j\omega + 5)(j\omega + 20)} \]
\[ |G(j\omega)|^2 = \frac{10^6}{\omega^2(\omega^2 + 25)(\omega^2 + 400)} = 1 \]
\[ \iff \omega^6 + 425\omega^4 + 10^4\omega^2 - 10^6 = 0 \]
\[ \omega^2 = -393.0887, \quad -68.8586, \quad 39.9456 \]
\[ \omega_{gc} = 6.078 \text{ rad/s} \]

Phase Margin

\[ PM = 180^\circ + \angle G(j\omega_c) = 180^\circ + \angle G(6.078) \]
\[ = 180^\circ - 157.4641^\circ \approx 22.54^\circ \]
Stability Margins

Delay Margin

$T_{dm} = \text{delay margin} = \text{time delay for the system to be on the verge of instability.}$

- Transfer function with time delay $T_d$
  \[ G(s)e^{-sT_d} \]

- System on verge of instability
  \[ G(j\omega_{gc})e^{-j\omega_{gc}T_{dm}} = -1 \]

Delay Margin Calculation

\[ G(j\omega_{gc})e^{-j\omega_{gc}T_{dm}} = -1 \]

- Equate angles:
  \[ \angle G(j\omega_{gc}) - \omega_{gc}T_{dm} \times \frac{180\degree}{\pi} = -180\degree \]
  \[ \angle G(j\omega_{gc}) = -180\degree + \text{PM} \]

- Solve for the delay margin
  \[ T_{dm} = \frac{\text{PM}}{\omega_{gc}} \times \frac{\pi}{180\degree} = \frac{180\degree + \angle G(j\omega_{gc})}{\omega_{gc}} \times \frac{\pi}{180\degree} \]

Example

\[ PM = 180\degree + \angle G(j\omega_{gc}) = 104\degree \]

\[ \omega_{gc} = 1.62 \text{ rad/s} \]

\[ T_{dm} = \frac{PM}{\omega_{gc}} \times \frac{\pi}{180\degree} = \frac{104\degree}{1.62} \times \frac{\pi}{180\degree} = 1.12 \]

Same as MATLAB answer.