What is the root locus?

- Plot of the loci of the closed-loop poles of a system as its gain is varied from 0 to $\infty$.
- Plots are rarely obtained for negative gains.
- Plots can be obtained for system parameters other than the gain (e.g. time constant).
- Useful design tool since closed-loop poles (and zeros) determine the time response.

System Configurations

- In both cases, use the loop gain for root locus plots.
- Only zeros of the forward path are closed-loop zeros.

Closed-loop Characteristic Equation

$$1 + KL(s) = 0$$

- Determined using the loop gain.

$$L(s) = G(s)H(s) = \frac{\prod_{i=1}^{n_z}(s - z_i)}{\prod_{j=1}^{n_p}(s - p_j)}$$

$z_i$ = open-loop zeros, $p_j$ = open-loop poles
Complex Equality

\[ KL(s) = -1 \]

Equivalent to two real equalities:

i. Magnitude condition

\[ K|L(s)| = 1 \]

ii. Angle Condition

\[ \angle L(s) = \pm (2m + 1)180^\circ, m = 0,1,2,\ldots \]

Complex Numbers

- Forms
  \[ x = x_R + jx_I = |x|e^{j\theta_x} \]

  Polar

- Cartesian

Arithmetic Operations

- Forms
  \[ x = x_R + jx_I = |x|e^{j\theta_x} \]

- Addition/Subtraction

  \[ x \pm y = (x_R \pm y_R) + j(x_I \pm y_I) \]

- Multiplication

  \[ x \times y = |x| \cdot |y|e^{j(\theta_x + \theta_y)} \]

- Division

  \[ x/y = |x|/|y|e^{j(\theta_x - \theta_y)} \]

Transfer Function

\[ G(s) = K \frac{s + a}{(s + b)(s + c)} \]

- Magnitude

  \[ |G(s)| = K \frac{|s + a|}{|s + b||s + c|} \]

- Angle

  \[ \angle G(s) = \angle(s + a) - \angle(s + b) - \angle(s + c) \]
Example: MATLAB

\[ G(s) = K \frac{(s + 3)(s + 4)}{(s + 1)(s + 2)} \]

Test Points: \(-2 + j3, \ -2 + j/sqrt(2)\) on root locus?

First test point

\[ g = \text{zpk}([-3, -4], [-1, -2], 1); \]

\[ \text{angle}(\text{evalfr}(g, -2 + j*3)) \]

\[ -1.2315 \]

- \[ -1.2315 \times 180/\pi = -70.56 \text{ degrees} \]
- \[ -2 + j3 \text{ not on root locus} \]

Second Test Point

Test Point: \(-2 + j/sqrt(2)\) on root locus?

\[ g = \text{zpk}([-3, -4], [-1, -2], 1); \]

\[ \text{angle}(\text{evalfr}(g, -2 + j/sqrt(2))) \]

\[ -3.1416 \]

\[ -2 + j/sqrt(2) \text{ on root locus} \]

\[ K = 1/\text{abs}(\text{evalfr}(g, -2 + j/sqrt(2))) \]

Calculator

\[ G(s) = K \frac{(s + 3)(s + 4)}{(s + 1)(s + 2)} \]

\[ s \leftarrow -2 + i/\sqrt{2} \]

\[ g = (s + 3) \times (s + 4)/(s + 1)/(s + 2) \]

- Angle: \(-180 \text{ degrees}\)
- On root locus
- Gain: \(K = 1/\text{abs}(g) = 1/3\)

Example

- Unity feedback closed-loop system with the forward gain:

\[ G(s) = \frac{K}{s^2} \]

- Closed-loop characteristic equation

\[ s^2 + K = 0, \quad s_{1,2} = \pm jK^{1/2} \]
Root Locus for Example 1

Root Locus: \( \angle G = \pm 90^\circ \pm 90^\circ = \pm 180^\circ \)

\[ g = \text{zpk([], [0,0], 1);} \]
\[ \text{rlocus}(g) \]

Numerical Computation

\[ L(s) = \frac{1}{s(s + 10)}, \quad 1 + KL(s) = 0 \]
\[ \Rightarrow s^2 + 10s + K = 0 \]
- Fix value of \( K \) and solve for the roots.

<table>
<thead>
<tr>
<th>( K )</th>
<th>Pole 1</th>
<th>Pole 2</th>
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<tbody>
<tr>
<td>0</td>
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<td>0</td>
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<td>30</td>
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<td>-5 + j3.16</td>
<td>-5 - j3.16</td>
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<td>-5 - j3.87</td>
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<td>-5 - j4.47</td>
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<tr>
<td>50</td>
<td>-5 + j5</td>
<td>-5 - j5</td>
</tr>
</tbody>
</table>