Root Locus Rules

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Outline

• Rules for sketching the root locus.
• Examples showing how the rules are applied.

Number of Root Locus Branches

1. The number of root locus branches (closed-loop poles) is equal to the number of open-loop poles of the loop gain $L(s)$.

$$1 + KL(s) = 0$$

$$L(s) = G(s)H(s) = \frac{\prod_{i=1}^{n_z} (s - z_i)}{\prod_{j=1}^{n_p} (s - p_j)}, n_p \geq n_z$$

$$K \prod_{i=1}^{n_z} (s - z_i) + \prod_{j=1}^{n_p} (s - p_j) = 0$$

Equation has $n_p$ roots.

Root Locus Branches

2. Branches start at the open-loop poles and end at the open-loop zeros or at infinity.

$$1 + KL(s) = 0, L(s) = \frac{\prod_{i=1}^{n_z} (s - z_i)}{\prod_{j=1}^{n_p} (s - p_j)}, n_p \geq n_z$$

$$K \prod_{i=1}^{n_z} (s - z_i) + \prod_{j=1}^{n_p} (s - p_j) = 0 \approx \begin{cases} \prod_{j=1}^{n_p} (s - p_j), K \to 0 \\ \prod_{i=1}^{n_z} (s - z_i), K \to \infty \end{cases}$$

$$K \frac{s^{n_z}}{s^{n_p}} \approx K s^{-(n_p - n_z)} \Rightarrow n_p - n_z \text{ zeros at } \infty$$
Example

How many branches? Start? End?

\[ G(s) = \frac{s + 1}{s(s + 2)(s + 5)} \]

- 3 root locus branches
- Start at: 0, -2, -5
- One branch ends at -1 (zero), two branches go to infinity
- \(3 - 1 = 2\) zeros at infinity

Real axis root loci

3- Real axis root loci have an odd number of real poles plus zeros to their right.
- Apply the angle condition to test points.
- No contribution from complex poles and zeros.

Example

Determine the real-axis loci \( G(s) = \frac{s + 1}{s(s + 2)(s + 5)} \)

Asymptotes

4- The branches going to infinity asymptotically approach the straight lines defined by the angle and intercept

\[ \theta_a = \pm \frac{(2m + 1)180^\circ}{n_p - n_z}, \quad m = 0, 1, 2, \ldots \]

\[ \sigma_a = \frac{\sum_{i=1}^{n_p} p_i - \sum_{j=1}^{n_z} z_j}{n_p - n_z} \]
**Example**

Find the asymptotes for \( G(s) = \frac{s+1}{s(s+2)(s+5)} \)

\[
\theta_a = \frac{\pm (2m+1)180^\circ}{n_p - n_z} = \frac{\pm (2m+1)180^\circ}{3-1} = \pm (2m+1)90^\circ, m = 0, 1, 2, \ldots
\]

\[
\sigma_a = \frac{\sum P_i - \sum z_j}{n_p - n_z} = \frac{(0 - 2 - 5) - (-1)}{3-1} = -\frac{3}{2}
\]

**Breakaway/Break-in Points**

5- Breakaway points (points of departure from the real axis) correspond to maxima of \( K \), while break-in points (points of arrival at the real axis) correspond to minima of \( K \).

- Find maxima (minima) using one of two approaches (on real axis \( K = -1/L(\sigma) \))
  
  i. \( \frac{dK}{d\sigma} = -\frac{d}{d\sigma} \left[ \frac{1}{L(\sigma)} \right] = 0 \)

  ii. Plot \( K = -\frac{1}{L(\sigma)}, i = 1, 2, \ldots \)

**Example**

Find the breakaway points for \( G(s) = \frac{s+1}{s(s+2)(s+5)} \)

\[ K := -\frac{1}{L(\sigma_i)}, i = 1, 2, \ldots \]

(ii) Plot \( K = -\frac{1}{L(\sigma_i)}, i = 1, 2, \ldots \)

**Graphical Approach**

- Peak at about 3.3

**MAPLE**

\[
g := \frac{s+1}{s(s+2)(s+5)}
\]

\[
K := -\frac{1}{L(\sigma_i)}
\]

\[
d := \frac{dK}{d\sigma} = -\frac{d}{d\sigma} \left[ \frac{1}{L(\sigma)} \right]
\]

\[
\text{diff(K, s)}
\]

\[
0.8203479570 + 0.9030131459 i, -3.359304086, -0.8203479570 - 0.9030131459 i
\]
Departure & Arrival Angles

6-The angle of departure from a complex pole \( p_n \) (angle of arrival at a complex zero \( z_m \)) is

\[
\theta_{p_n} = \mp 180^\circ + \angle G_{p_n}(p_n)
\]

\[
G_{p_n}(s) = G(s)(s - p_n)
\]

\[
\angle G(s) = \angle G_{p_n}(p_n) - \angle(s - p_n)
\]

\[
\theta_{z_m} = \pm 180^\circ - \angle G_{z_m}(z_m)
\]

\[
G_{z_m}(s) = G(s)/(s - z_m)
\]

\[
\angle G(s) = \angle G_{z_m}(z_m) + \angle(s - z_m)
\]

Proof of Rule

\[
L(s) = \prod_{i=1}^{n} \left( \frac{s - z_i}{s - p_n} \right) = \frac{1}{\prod_{j=1}^{m} (s - p_j)}
\]

\[
L(s_{cl}) = (s - z_n) \times G_{n_p}(s_{cl}) = \frac{1}{s - p_{n_p}} \times G_{n_p}(s_{cl})
\]

- Apply the angle condition to test points close to the complex pole (zero).

\[
\pm \pi = \theta_{z_m} + \angle G_{z_m}(z_m) \Rightarrow \theta_{z_m} = \pm \pi - \angle G_{z_m}(z_m)
\]

\[
\pm \pi = -\theta_{p_n} + \angle G_{n_p}(p_n) \Rightarrow \theta_{p_n} = \mp \pi + \angle G_{n_p}(p_n)
\]

Example

Sketch the root locus for the system

\[
G(s) = \frac{1}{(s + 4)(s^2 + 4s + 8)}
\]

\[
G(s) = \frac{1}{(s + 4)(s + 2 - j2)(s + 2 + j2)}
\]

\[
\theta_{p_n} = \mp 180^\circ + \angle G_{p_n}(p_n)
\]

\[
\theta_{p_2} = 180^\circ + \angle \left[ \frac{1}{(s + 4)(s + 2 + j2)} \right]_{s=-2+j2}
\]

\[
= 45^\circ
\]

See root locus

Root Locus for Example
Example 8.5 (Nise)

\[ G(s) = \frac{s + 3}{s(s + 1)(s + 2)(s + 4)} \]

Closed-loop Transfer Function

\[ T(s) = \frac{K N(s)}{K N(s) + D(s)} = \frac{K(s + 3)}{s^3 + 7s^2 + 14s^2 + (8 + K)s + 3K} \]

- Third row multiplied by 7 to simplify calculations.
- For stability \( K > 0, K < 90 \), and the quadratic is positive
- The quadratic has the roots \( K_1 = -74.646, K_2 = 9.6465 \)

Multiple Constraints

- Negative 2nd derivative of quadratic (negative \( K^2 \) coefficient): maximum (concave)
- Quadratic is positive for \( K_1 < K < K_2 \)
- Stable range=intersection of \( K \) ranges for positive entries of the first column
  \[ 0 < K < K_2 = K_{cr} = 9.6456 \]
- Auxiliary Equation: \((90 - K_{cr})s^2 + 21K_{cr} = 0\)
  \((90 - 9.6456)s^2 + 21 \times 9.6456 = 0\)
- Intersections of RL with \( j\omega \)-axis:
  \[ s_{12} = \pm j1.5877 \]

Conclusion

- Apply rules to sketch root locus.
- Some rules may not be applicable.
- MATLAB: numerical solution for roots without the use of rules for root locus sketching.