Lag-lead Design
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When is it needed?
• Proportional control does not meet specifications (both transient response and steady-state error).
• Lead meets transient response specifications.
• Need a lag section to reduce the steady-state error.

General form
• Lag-lead compensator

\[
C(s) = K \left( \frac{T_{\text{lead}} s + 1}{\alpha T_{\text{lead}} s + 1} \right) \left( \frac{T_{\text{lag}} s + 1}{\beta T_{\text{lag}} s + 1} \right)
\]

\[
= \left( \frac{K}{\alpha \beta} \right) \left( \frac{s + 1 / T_{\text{lead}}}{s + 1 / \alpha T_{\text{lead}}} \right) \left( \frac{s + 1 / T_{\text{lag}}}{s + 1 / \beta T_{\text{lag}}} \right)
\]

\[
\alpha < 1, \quad \beta > 1
\]

Design Options
1. \( \alpha \beta \neq 1 \)

\[
C(s) = K \frac{T_{\text{lead}} s + 1}{\alpha T_{\text{lead}} s + 1} \times \frac{T_{\text{lag}} s + 1}{\beta T_{\text{lag}} s + 1}
\]

2. \( \alpha \beta = 1 \)

\[
C(s) = K \frac{T_{\text{lead}} s + 1}{\alpha T_{\text{lead}} s + 1} \times \frac{T_{\text{lag}} s + 1}{\alpha^{-1} T_{\text{lag}} s + 1}
\]

i. Select a suitable value for \( \alpha \).
ii. Design procedure (skip).
1. $\alpha \beta \neq 1$

- Design a lead compensator to improve the transient response.
- Improve the steady-state response by adding a lag compensator.
- Because the lag compensator will cause a deterioration in the transient response, it is advisable to use a somewhat conservative lead design.

**Example**

Design a lag-lead compensator for the system to obtain closed-loop dominant poles with a damping ratio $\zeta$ of 0.5, an undamped natural frequency $\omega_n$ of 4 rad/s, and reduce the steady-state error by a factor of 5.

$$L(s) = \frac{4}{s(s + 2)}$$

**Solution $\alpha \beta \neq 1$: Lead Design**

From the lead presentation, we have

$$s_{cl} = -2 + j3.464$$

and the lead compensator

$$G_{lead}(s) = 4.7 \frac{s + 2.9}{s + 5.5} \approx 2.5 \frac{s/2.9 + 1}{s/5.5 + 1}$$

**Solution $\alpha \beta \neq 1$: Lag Design**

Type 1: The lead section increases $K_v$ by 2.5 so we need $\beta = 2$

$$G_{Lag}(s) = \frac{s + 0.2}{s + 0.1} = 2 \frac{5s + 1}{10s + 1}$$

$$C(s) = 5 \frac{s/2.9 + 1}{s/5.5 + 1} \times \frac{5s + 1}{10s + 1}$$

Check the time response and, if necessary, adjust the design to meet the specs.
2.1 $\alpha \beta = 1$

- Choose a value for $\alpha = 1/\beta$ sufficiently small to provide the desired specifications.
- $\beta$ provides the necessary improvement in steady-state error.
- Design a lead compensator with the selected $\alpha$.
- For a conservative design, use $\alpha = 0.1$ and lead compensator angle
  \[ \phi = -180^\circ - \angle L(s_{cl}) + 3 - 5^\circ \]
- For pole-zero cancellation (cancel pole at $-p$)
  \[ G_{\text{Lead}}(s) = \frac{s + a}{s + b} = \frac{s + p}{s + p/\alpha} \]

Example

Design a lag-lead compensator for the system to obtain closed-loop dominant poles with a damping ratio $\zeta$ of 0.5, an undamped natural frequency $\omega_n$ of 4 rad/s, and reduce the steady-state error by a factor of 3. Use the passive circuit subject to the constraint $\beta = 1/\alpha$

\[ L(s) = \frac{4}{s(s + 2)} \]
Solution

- Calculate the desired closed-loop pole location

\[ g = \text{zpk([], [0, -2], 4); scl = -2 + j \times 2 \times \sqrt{3}} \]

\[ \text{scl} = -2.0000 + 3.4641i \]

- The angle of the loop gain at the desired closed-loop pole location is

\[ \phi = \text{pi-angle(evalfr(g, scl))} \]

\[ \phi = 0.5236 \]

Can use to calculate pole and zero locations.

Solution (Page 2) Easier Design

- Try \( \alpha = 0.2, \beta = 5 \). Meets design specifications.

- Cancel pole with zero: zero at \(-2\)

- Compensator pole at \(-2/\alpha = -10\)

- MATLAB root locus plot: Gain = 24.9 for \( \zeta = 0.5 \)

- Lag Compensator:

\[ G_c(s) = \frac{24.6(s + 2)}{s + 10} \times \frac{s + 0.5}{s + 0.1} \]

Root Locus: Lead Compensation

MATLAB Commands

\[ \alpha = 0.2; \% \alpha = 1/\beta \]

\[ a = \text{real(scl)/10;} \]

\[ gc = \text{zpk(-2, -2/alpha, 1) \times zpk(-a, -alpha*a, 24.6);} \]

Zero/pole/gain:

24.6 \((s+2)\) \((s+0.5)\)

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\((s+10)\) \((s+0.1)\)
Compensated System
Adjust gain to improve the transient response: Reduce gain to 12 (compromise between settling and overshoot).

2.ii $\alpha \beta = 1$: Design Procedure
- Calculate the error constant for the system with proportional control and calculate the needed gain $K$ to meet the steady-state error requirements.
- Obtain the ratio $r$ of the pole distance to the zero distance for the lead section using
  \[ r = \frac{d_p}{d_z} = K|L(s_{cl})| \]
  \[ d_z = |s_{cl} + z|, \quad d_p = |s_{cl} + p| \]

2.ii $\alpha \beta = 1$ Proc. (Page 2)
- Calculate the angle contribution of the lead section $\phi = -180^\circ - \angle L(s_{cl}) + 3 - 5^\circ$.
- Obtain the pole and zero locations for the lead section using
  \[ \frac{1}{\tan(\theta_z)} = \frac{\cos(\phi) - r^{-1}}{\sin(\phi)} \]
  \[ z = \zeta \omega_n + \frac{\omega_d [\cos(\phi) - r^{-1}]}{\sin(\phi)} \]
  \[ \frac{1}{\tan(\theta_p)} = \frac{r - \cos(\phi)}{\sin(\phi)} \]
  \[ p = \zeta \omega_n + \frac{\omega_d [r - \cos(\phi)]}{\sin(\phi)} \]

2.ii $\alpha \beta = 1$ Proc. (Page 3)
- Add a lag section with $\beta = 1/\alpha = p/z$ where $p$ and $z$ are the pole and zero of the lead section.
- Check the error constant and the pole locations for the design.
- Check the time response and modify the design if necessary to meet the design specifications.