## Review of Classical Control

M. Sami Fadali  
Professor of Electrical Engineering  
University of Nevada

---

### Laplace Transform

- Transforms differential equations to algebraic equations.  
- Transforms convolution to multiplication and allows us to use transfer functions.  
- Used with linear time-invariant (LTI) systems. 

\[
F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st} dt \\
\quad s = \sigma + j\omega
\]

### Important Laplace Transforms

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Impulse ( \delta(t) )</td>
<td>1</td>
</tr>
<tr>
<td>Unit Step ( 1(t) )</td>
<td>( 1/s )</td>
</tr>
<tr>
<td>Unit Ramp ( t )</td>
<td>( 1/s^2 )</td>
</tr>
<tr>
<td>Sinusoid ( \sin(\omega_0 t) )</td>
<td>( \omega_0/(s^2 + \omega_0^2) )</td>
</tr>
<tr>
<td>Sinusoid ( \cos(\omega_0 t) )</td>
<td>( s/(s^2 + \omega_0^2) )</td>
</tr>
<tr>
<td>( e^{-\zeta \omega_n t} \sin(\omega_d t) )</td>
<td>( \omega_d/(s + \zeta \omega_n)^2 + \omega_d^2 )</td>
</tr>
</tbody>
</table>

---

### 4.3 First Order Systems

- Differential Equation \( \dot{c}(t) + a \ c(t) = a \ r(t) \)

- Transfer Function

\[
\frac{C(s)}{R(s)} = G(s) = \frac{a}{s + a} = \frac{1}{\tau s + 1}, \ a = \frac{1}{\tau}
\]

- Unit Step Response

\[
c(t) = 1 - e^{-at} = 1 - e^{-t/\tau}, \ t \geq 0
\]
Specifications for 1st Order Systems

- **Rise Time**: time to go from 10% to 90% of the final value.
  \[ T_r \approx 2.2 \tau \]
- **Settling Time**: time to reach and stay within a specified percentage of the final value.
  \[ T_s = 4\tau \quad (2\%) \]
  \[ T_s = 3\tau \quad (5\%) \]
  \[ T_s = 5\tau \quad (1\%) \]

### 4.4 Second Order Systems

**Differential Equation**
\[ \ddot{c} + 2\zeta \omega_n \dot{c} + \omega_n^2 c = \omega_n^2 r \]

**Transfer Function**
\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2)} \]

**Unit Step Response**
\[ C(s) = G(s)R(s) = \frac{\omega_n^2}{(s^2 + 2\zeta \omega_n s + \omega_n^2)s} \]
\[ = \frac{1}{s} + \frac{A_1 s + A_2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

**Important Terms**
- \( \omega_n \) = undamped natural frequency rad/s
- \( \zeta \) = damping ratio
- \( \omega_d \) = damped natural frequency rad/s
\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]
Step Response of 2nd Order System

<table>
<thead>
<tr>
<th>Time (sec.)</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

2nd Order Identities

- Settling Time \( T_s = \frac{4}{\zeta \omega_n} \)
- Peak Time \( T_p = \frac{\pi}{\omega_d} \)
- Overshoot \( \%OS = \exp\left(-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}\right) \times 100\% \)
- Damping ratio \( \zeta = \frac{x}{\sqrt{x^2 + \pi^2}}, \ x = \left| \ln\left(\frac{\%OS}{100}\right) \right| \)

Dominant 2nd Order Pair

- A 2nd order pair with zeros or additional poles that are all located far in the LHP.
- Time response is approximately the same as that of the 2nd order pair.

Example: poles \( s_{1,2} = -10 \pm j8, \ s_3 = -50 \)

The underdamped pair is dominant since \( \exp(-50t) \) decays to zero quickly.

- RULE of THUMB factor of 5 is enough.

The Final Value Theorem

- If a function approaches a constant limit as \( t \) tends to infinity, then the limit is given by

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]

- Cases where limit does not exist
  (i) An unbounded function.
  (ii) An oscillatory function.
Steady-state Error

\[ E(s) = R(s) - C(s) = [1 - T(s)]R(s) \]

\[ e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} [1 - T(s)]sR(s) \]

- Use for non-unity feedback systems.

Unity Feedback Error

<table>
<thead>
<tr>
<th>Input</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1(t)</td>
<td>1/(K_p+1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ramp t</td>
<td>\infty</td>
<td>1/K_v</td>
<td>0</td>
</tr>
<tr>
<td>Parabolic t^2/2</td>
<td>\infty</td>
<td>\infty</td>
<td>1/K_a</td>
</tr>
</tbody>
</table>

Root Locus

- Locus of closed-loop poles as gain \( K \) varies between 0 and \( \infty \).
- Magnitude and phase conditions based on the closed-loop characteristic equation

\[ 1 + K L(s) = 0 \]

\( L(s) = \text{open-loop gain} \)

Effect of Zero
**Frequency Response**

Linear System

\[ G(j\omega_0) = M(\omega_0)e^{j\phi(\omega_0)} \]

Input \( r(t) = A\sin(\omega_0 t) \)

Output \( c(t) = AM(\omega_0)\sin(\omega_0 t + \phi(\omega_0)) \)

![Block Diagram](image)

**Frequency Response Plots**

- Obtain using sinusoidal inputs.
- Plot complex numbers vs. frequency.
- Three standard frequency response plots
  1. Polar (Nyquist)
  2. Bode
  3. Log Magnitude vs. Phase (Nichols)

**3rd Order Type 0 System with 2nd Order Underdamped Pair**

![Nyquist Diagram](image)

**Nyquist Stability Criterion**

\[ F(s) = 1 + L(s) \]

\[ = 1 + \frac{N_L(s)}{D_L(s)} = \frac{N_L(s) + D_L(s)}{D_L(s)} \]

- Zeros of \( F(s) \) are closed-loop poles
- Poles of \( F(s) \) are open-loop poles
Nyquist Contour

- Encloses all c.l. and o.l. poles in RHP.
- Plot \( G(s) \) for \( s \) on contour traversed clockwise.
- Modify contour for \( j\omega \)-axis poles.
- \( Z \) = no. of closed-loop poles in the RHP.
- \( P \) = no. of open-loop poles in the RHP.

Nyquist Criterion (a)

- Unstable system: number of closed-loop poles in the RHP = \( Z = P - N \)
- \( N \) = No. of counterclockwise encirclements of \((-1,0)\) by \( L(s) \)
- \( P \) = No. of open-loop poles of \( L(s) \) in RHP
- Necessary and sufficient condition for the closed-loop stability of a system with loop gain \( L(s) \) is \( N = P \)

Nyquist Criterion (b)

For open-loop stable systems \( (P = 0) \)
(a) Closed-loop stable in the absence of \((-1,0)\) encirclements.
(b) Closed-loop unstable system:
    - number of RHP poles = \(-N\)
    - number of clockwise encirclements of \((-1,0)\)

Simplified Nyquist Criterion

Open-loop stable system \( (P = 0) \).
- Observer follows the polar plot of the loop gain in the direction of increasing frequency.
- Closed-loop system is stable if and only if the point \((-1,0)\) is to the left of the observer when crossing the real axis.

Remark: The criterion implies no encirclements of \((-1,0)\) by the complete Nyquist plot.
Gain Margin and Phase Margin

Nyquist Diagram

Real Axis

Imaginary Axis

Gain Margin and Phase Margin

Nyquist Diagram

Real Axis

Imaginary Axis

PM & GM: Bode Plot

Bode Diagram

Gm = 7.6 dB (at 4.47 rad/s) , Pm = 19.9 deg (at 2.8 rad/s)

PM & GM: Bode Plot

Bode Diagram

Gm = 7.6 dB (at 4.47 rad/s) , Pm = 19.9 deg (at 2.8 rad/s)

>> s=tf('s'); g=100/(s+2)/(s+10)

Change Properties

Bode Diagram

Gm = 2.4  (at 4.47 rad/s) ,  Pm = 19.9 deg (at 2.8 rad/s)

Change Properties

Bode Diagram

Gm = 2.4  (at 4.47 rad/s) ,  Pm = 19.9 deg (at 2.8 rad/s)

System: g

Frequency (rad/s): 0.913

Magnitude (abs): 4.95

System: g

Frequency (rad/s): 4.49

Magnitude (abs): 0.413

Nyquist Plot

Nyquist Diagram

System: g

Gain Margin (dB): 7.6
At frequency (rad/s): 4.47
Closed Loop Stable? Yes

Nyquist Plot

Nyquist Diagram

System: g

Gain Margin (dB): 7.6
At frequency (rad/s): 4.47
Closed Loop Stable? Yes