Basis Set
- Use a new basis set for state space.
- Obtain the state-space matrices for the new basis set.
- Similarity transformation.

Transformation
- Given a state vector $x(k)$, define a new state vector $z(k)$ using the invertible transformation matrix $T$:
  $$ x(k) = Tz(k) \iff z(k) = T^{-1}x(k) $$
- Substitute for $x(k)$:
  $$ x(k + 1) = Ax(k) + Bu(k) $$
  $$ Tz(k + 1) = ATz(k) + Bu(k) $$
  $$ y(k) = CTz(k) + Du(k) $$
- Premultiply the state equation by $T^{-1}$:
  $$ z(k + 1) = T^{-1}ATz(k) + T^{-1}Bu(k) $$

Realization
- State-space quadruple for the state vector $z(k)$:
  $$(A, B, C, D) = (T^{-1}AT, T^{-1}B, CT, D)$$
- The quadruple for the state vector $x(k)$ can be obtained from the quadruple of $z(k)$ using the inverse transformation $T^{-1}$.
- Similarity transformation is identical for continuous-time and discrete-time systems.
- Continuous-time system state vector $z(t)$:
  $$ x(t) = Tz(t) \iff z(t) = T^{-1}x(t) $$
**ALTERNATIVE EXPRESSION**

(A, B, C, D) = \((T^{-1}AT, T^{-1}B, CT, D)\)
- Rewrite the matrices in terms of the matrix \(S = T^{-1}\)
  \((A, B, C, D) = (SAS^{-1}, SB, CS^{-1}, D)\)
- Equivalent to the first expression.
- Used in MATLAB
- MATLAB command for similarity transformation
  » \(Pt = \text{ss2ss}(P, S)\)

**TRANSFORMATION TO DIAGONAL FORM**

\((A_x, B_x, C_x, D) = (T^{-1}AT, T^{-1}B, CT, D)\)

- For \(T = V\)
  \(A_x = V^{-1}AV = V^{-1}(V\Lambda V^{-1})V = \Lambda\)
  \(B_x = V^{-1}B, C_x = CV, D\)
  \(V = \text{modal matrix of eigenvectors of } A\)
  \(\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \lambda_i = \text{eigenvalues of } A\)
  \(T = V = \text{modal transformation}\)
- Recall
  \(e^{\Lambda t} = \text{diag}\{e^{\lambda_1t}, e^{\lambda_2t}, \ldots, e^{\lambda_nt}\}\)
  \(\Lambda^k = \text{diag}\{\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k\}\)

**INVERSE OF MODAL MATRIX**

Invert of Modal Matrix

\[
W = V^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}
\]

Input matrix

\[B_x = V^{-1}B = WB = \begin{bmatrix} w_1^T B \\ \vdots \\ w_n^T B \end{bmatrix}\]

Output matrix

\[C_x = CV = C[v_1 \ldots v_n] = [Cv_1 \ldots Cv_n]\]

**EXAMPLE 7.18**

Obtain the diagonal form for the state-space equations

\[
\begin{bmatrix}
x_1(k+1) \\
x_2(k+1) \\
x_3(k+1)
\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.04 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)
\]

\[y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}\]
**SOLUTION**

- **eig command of MATLAB**
  
  \[ \Lambda = \text{diag}\{0, -0.1, -0.4\} \]
  
  \[ V = \begin{bmatrix} 1 & -0.995 & 0.9184 \\ 0 & 0.0995 & -0.36741 \\ 0 & -0.00995 & 0.1469 \end{bmatrix} \]

**COMPANION FORM**

- The state matrix is in companion form and the modal matrix is the *Van der Monde* matrix

\[
V = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda^2_1 & \lambda^2_2 & \lambda^2_3 \\ \lambda^3_1 & \lambda^3_2 & \lambda^3_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -0.4 \\ 0 & 0.01 & 0.16 \end{bmatrix}
\]

MATLAB command for similarity transformation

- \[ Pt = \text{ss2ss}(P, \text{inv}(V)) \]

**DIAGONAL FORM**

- The commands yields

\[ At = \text{diag}\{0, -0.1, -0.4\} \]

\[ Bt = \begin{bmatrix} 25 \\ 33.5012 \\ 9.0738 \end{bmatrix} \]

\[ Ct = \begin{bmatrix} 1.0000 -0.9950 & 0.9184 \end{bmatrix} \]

- **Transfer function**: can be written by inspection since the system is given in controllable form.

- It can be easily verified that the above diagonal form is different from that obtained using partial fraction expansion of the system transfer function (covered later).

**INVARINANCE OF TF & CHARACTERISTIC EQUATION**

- **Theorem 7.1**: Similar systems have identical transfer functions and characteristic polynomial.
**PROOF: CHARACTERISTIC POLYNOMIAL**

- Characteristic polynomial of \((A_1, B_1, C_1, D)\)
  \[
  \det[zI_n - A_1] = \det[zI_n - T^{-1}AT] \\
  = \det[T^{-1}(zI_n - A)T] \\
  = \det[T^{-1}] \det[zI_n - A] \det[T] \\
  = \det[zI_n - A]
  \]

- Characteristic polynomial of \((A, B, C, D)\)
- Used the identity
  \[
  \det[T^{-1}] \det[T] = \det[I_n] = 1
  \]

**PROOF 2: TRANSFER FUNCTION**

- Transfer function matrix of \((A_1, B_1, C_1, D)\)
  \[
  G_1(s) = C_1[zI_n - A_1]^{-1}B_1 + D \\
  = CT[zI_n - T^{-1}AT]^{-1}T^{-1}B + D \\
  = C[T(zI_n - T^{-1}AT)T^{-1}]^{-1}B + D \\
  = C[zI_n - A]^{-1}B + D = G(s)
  \]

- Transfer function matrix of \((A, B, C, D)\)
- Used the identity \((A B C)^{-1} = C^{-1} B^{-1} A^{-1}\).