Stability

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Outline

• Asymptotic stability.
• BIBO stability.
• Examples.

Equilibrium

• An equilibrium point or state is an initial state from which the system never departs unless perturbed.
• Nonlinear system \( x(k + 1) = f[x(k)] \)
  \( f[x_e] = x_e \)
• System may have multiple equilibrium points
• Linear system \( x(k + 1) = Ax(k) \)
  \( Ax_e = x_e \iff [A - I_n]x_e = 0 \)
• Unique equilibrium \( x_e = 0 \) if \([A - I_n]\) is nonsingular.

Asymptotic Stability

A linear system is asymptotically stable if all its trajectories converge to the origin
i.e. for any initial state \( x(k_0) \),
\( x(k) \to 0 \) as \( k \to \infty \).
• Also called \textbf{internal stability}.
• Define stability for the system with equilibrium at the origin.
• For nonlinear systems, define stability of an equilibrium point.
Asymptotic Stability Condition

**Theorem 8.1**

A discrete-time linear system is asymptotically (Schur) stable if and only if all the eigenvalues of its state matrix are inside the unit circle.

\[ |\lambda_i(A_d)| < 1, \ i = 1,2, \ldots, n \]

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**Proof**

Zero-input response

\[ x_{Z1}(k) = A^{k-k_0}x(k_0) = \sum_{i=1}^{n} Z_i x(k_0) \lambda_i^{k-k_0} \]

**Sufficiency:** \( x(k) \to 0 \) as \( k \to \infty \) if all \( \lambda_j \) inside unit circle.

**Necessity:**

Let \( A \) be stable with eigenvalue \( \lambda_j \) outside unit circle.

System Response for \( x(k_0) = v_j = j^{th} \) eigenvector of \( A \)

\[ x_{Z1}(k) = \sum_{i=1}^{n} Z_i x(k_0) \lambda_i^{k-k_0} = \sum_{i=1}^{n} v_i w_i^T v_j \lambda_i^{k-k_0} = v_j \lambda_j^{k-k_0} \]

unbounded as \( k \to \infty \). Contradiction.

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**Remarks**

**Necessary Stability Condition:**

Convergence to 0 for all initial conditions.

\[ Z_i v_j = 0, \ i \neq j, \quad Z_j v_j = v_j \]

For \( |\lambda_j(A_d)| \geq 1 \), not all trajectories converge to the origin.

- **Necessary Stability Condition:** Invertible matrix \( A - I_n \).

\[ A - I_n = V[A - I_n]V^{-1} \]

\[ Vdiag{\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1}V^{-1} \]

- Stable systems have a unique equilibrium at 0

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**Example 8.1,2**

Find the equilibrium points of the systems and determine the stability of (b) and (c).

a) \( x(k + 1) = x(k) [x(k) - 0.5] \)

b) \( x(k + 1) = 2x(k) \)

c) \[ \begin{bmatrix} x_1(k + 1) \\ x_2(k + 1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \]
Solution

a) Equilibrium condition: \( x_e = x_e [x_e - 0.5] \)
\( x_e [x_e - 1.5] = 0 \)
Two equilibrium states: \( x_e = 0, x_e = 1.5 \)

b) Equilibrium condition: \( x_e = 2x_e \)
One equilibrium state: \( x_e = 0 \).

c) One equilibrium state: \( x_e = [x_{1e} \ x_{2e}]^T = 0^T. \)
\[
\begin{bmatrix}
[x_{1e}]
\end{bmatrix} = \begin{bmatrix}
0.1 & 0
\end{bmatrix}
\begin{bmatrix}
[x_{2e}]
\end{bmatrix}
\]
\[
\Rightarrow \begin{bmatrix}
0.1 - 1 & 0
1 & 0.9 - 1
\end{bmatrix}
\begin{bmatrix}
[x_{1e}]
\end{bmatrix} = \begin{bmatrix}0
\end{bmatrix}
\]

Stability Determination 8.1(b)

- Response due to any initial state \( x(0) \)
  \( x(1) = 2x(0) \)
  \( x(2) = 2x(1) = 2 \times 2x(0) = 2^2 x(0) \)
  \( x(3) = 2x(2) = 2 \times 2^2 x(0) = 2^3 x(0) \)
  \( \vdots \)
  \( x(k) = 2^k x(0) \)
  Unbounded as \( k \to \infty \) (unstable).
- The system has one eigenvalue at \( 2 > 1 \),
  which violates the stability condition of Theorem 8.1.

Stability Determination 8.1(c)

\[
\begin{bmatrix}
[x_1(k + 1)]
[x_2(k + 1)]
\end{bmatrix} = \begin{bmatrix}
0.1 & 0
1 & 0.9
\end{bmatrix}
\begin{bmatrix}
[x_1(k)]
[x_2(k)]
\end{bmatrix}
\]

System response due an arbitrary initial state \( x(0) \)
\[
\begin{bmatrix}
[x_1(k)]
[x_2(k)]
\end{bmatrix} = \begin{bmatrix}
1
-1/0.8
\end{bmatrix}
\begin{bmatrix}
0
0
\end{bmatrix}
(0.1)^k[x_1(0)]
\]
\[
+ \begin{bmatrix}
0
1/0.8
\end{bmatrix}
\begin{bmatrix}
0
0
\end{bmatrix}
(0.9)^k[x_2(0)]
\]

- Response decays to zero as \( k \to \infty \)
  (asymptotically stable).
- Both system eigenvalues (0.1 and 0.9) are inside
  the unit circle i.e. satisfy the conditions of
  Theorem 8.1.

Input-output (BIBO) Stability

Definition For any bounded input, the output is bounded.
\( \|u(k)\| < k_u, \forall k \Rightarrow \|y(k)\| < k_y, \forall k, \forall u(k) \)

- For SISO systems replace \( \|.\| \) (norm) with
  \( |.| \) (absolute value)
- Note: the condition must hold for any input
  and for all time \( k \)
- Definition can be generalized to distributed parameter systems.
Theorem 1: BIBO Stability

A SISO LTI system is BIBO stable if and only if its impulse response satisfies

$$\sum_{k=0}^{\infty} |g(k)| < \infty$$

Remarks

- Condition can be generalized to time-varying MIMO systems using ||.|| (norm) in place of |.|.
- Condition can be generalized to distributed parameter systems.

Proof Thm. 8.2 BIBO Stability

- Sufficiency (if) $|u(k)| < k_u, \forall t$

  $$y(k) = \sum_{i=0}^{\infty} g(i)u(k-i)$$

  $$|y(k)| \leq \sum_{i=0}^{\infty} |g(i)| |u(k-i)| < k_u \sum_{i=0}^{\infty} |g(i)| < \infty$$

  Necessity (only if): Assume BIBO stable with condition violated and let

  $$u(k-i) = \text{sign}[g(i)], |u(i)| \leq 1, \forall t$$

  $$y(t) = \sum_{i=0}^{\infty} g(i)u(k-i) \leq \sum_{k=0}^{\infty} |g(k)| > \infty \text{ Contradiction.}$$

Theorem 8.3: BIBO Stability

LTI SISO system is BIBO stable if and only if all its transfer function poles are inside the unit circle.

Proof (Necessity)$\sum_{i=0}^{\infty} |g(i)| = |d| + \sum_{i=1}^{\infty} |g(i)|$

$$= |d| + \sum_{i=0}^{\infty} \sum_{j=1}^{n} |CZ_j B \lambda_j^i| \leq |d| + \sum_{i=0}^{\infty} \sum_{j=1}^{n} |CZ_j B| |\lambda_j^i|$$

$$\leq |d| + \sum_{j=1}^{n} |CZ_j B| \sum_{i=0}^{\infty} |\lambda_j^i| \leq |d| + \sum_{j=1}^{n} |CZ_j B| \sum_{i=1}^{\infty} |\lambda_j^i|$$

The last summation is bounded only if any term for which $CZ_j B$ is nonzero, $|\lambda_j| < 1$

BIBO Stability

- Poles of z-transfer function inside the unit circle.
- Asymptotic stability implies BIBO stability.
- For a minimal system (not in general), BIBO stability implies asymptotic stability (no pole-zero cancellation).
Example 8.4

Test the BIBO stability of the system

\[ G(z) = \frac{\begin{bmatrix} z & -0.1 \\ -0.9 & z \end{bmatrix}}{z(z - 0.3)(z + 0.3)} \]

Solution

- All poles are inside the unit circle.
- The system is BIBO stable.