Outline

- Model of ADC.
- Model of DAC.
- Model of ADC, analog subsystem and DAC.
- Systems with transport lag.
- Examples

ADC Model

Assume that
(i) ADC outputs are exactly equal in magnitude to their inputs. (i.e. neglect quantization errors).
(ii) ADC yields digital output instantaneously.
(iii) Sampling is perfectly uniform.
Model ADC as ideal sampler with sampling period $T$. 

Common Digital Control System Configuration

- Computer or Microprocessor
- DAC
- Analog Subsystem
- ADC

Diagram:

- $T$
DAC Model
Assume that DAC
(i) outputs are exactly equal in magnitude to their inputs.
(ii) yields an analog output instantaneously.
(iii) outputs are constant over each sampling period.
zero-order-hold: piecewise constant analog output.
first-order hold constructs signals in terms of straight lines
second-order hold constructs them in terms of parabolas.
\{u(k)\} \rightarrow u(t) = u(k), kT \leq t < (k + 1)T, k = 0, 1, 2, ...

Transfer Function of ZOH
Use impulse sampling and Laplace transform of a unit pulse
\{1(t)\} = \frac{1}{s} \quad \{1(t - T)\} = \frac{e^{-sT}}{s}
G_{ZOH}(s) = \frac{1 - e^{-sT}}{s}

Effect of Sampler on Transfer Function of a Cascade
In the s-domain
\[ Y(s) = H_2(s)X(s) = H_2(s)H_1(s)U(s) \]
\[ = H_{eq}(s)U(s), \quad H_{eq}(s) = H_2(s)H_1(s) \]

Time Response
\[ y(t) = \int_{0}^{t} h_2(t - \tau)x(\tau)d\tau \]
\[ = \int_{0}^{t} h_2(t - \tau)\left[ \int_{0}^{\tau} h_1(\tau - \lambda)u(\lambda)d\lambda \right]d\tau \]
Change order and variables of integration
\[ y(t) = \int_{0}^{t} u(t - \tau)\left[ \int_{0}^{\tau} h_1(\tau - \lambda)h_2(\lambda)d\lambda \right]d\tau \]
\[ = \int_{0}^{t} u(t - \tau)h_{eq}(\tau)d\tau \]
**Punch Line**

Equivalent impulse response for cascade

= the convolution of cascaded impulse responses.

- Impossible to separate convolved time function

\[ h_{eq}(t) = \int_0^t h_1(t - \lambda)h_2(\lambda)d\lambda \]

- Sample the output

\[ y(iT) = \int_0^{iT} u(iT - \tau)h_{eq}(\tau)d\tau, \quad i = 0, 1, 2, \ldots \]

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**Convolving \( u^*(t) \) & CT Signal**

\[ y(t) = \int_0^t h(t - \tau)u(\tau)d\tau \]

\[ = \int_0^t h(t - \tau) \left[ \sum_{k=0}^{\infty} u(kT) \delta(\tau - kT) \right] d\tau \]

Repetitions of the CT function each in the location of an impulse in the train

\[ \begin{array}{c}
U(s) \xymatrix{ \times \ar[r] & U'(s) } \\
\ar[r] & H(s) \ar[r] & Y(s)
\end{array} \]

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**Sampled Output**

- Change the order of summation and integration

\[ y(t) = \sum_{k=0}^{\infty} u(kT) \int_0^t h(t - \tau)\delta(\tau - kT)d\tau \]

\[ = \sum_{k=0}^{\infty} u(kT)h(t - kT) \]

- Sample and transform

\[ y(iT) = \sum_{k=0}^{\infty} u(kT)h(iT - kT), \quad i = 0, 1, 2, \ldots \]

\[ Y(z) = H(z)U(z), \quad Y^*(s) = H^*(s)U^*(s) \]

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**Transfer Function for Cascade**

- **Cascade not separated by samplers**: components cannot be separated after sampling.

- **Cascade separated by samplers**: each block has a sampled output and input as well as a z-domain transfer function.

- \( n \) blocks not separated:

\[ Y(s) = H(z)U(z) = (H_1H_2 \ldots H_n)(z)U(z) \]

- \( n \) blocks separated:

\[ Y(s) = H(z)U(z) = H_1(z)H_2(z) \ldots H_n(z)U(z) \]
Example
Find the equivalent sampled impulse response sequence and the equivalent z-transfer function for the cascade of the two analog systems with sampled input
(a) If the systems are directly connected.
(b) If the systems are separated by a sampler.
\[ H_1(s) = \frac{1}{s+2}, \quad H_2(s) = \frac{2}{s+4} \]

(a) Systems Directly Connected
\[ H(s) = \frac{2}{(s+2)(s+4)} = \frac{1}{s+2} - \frac{1}{s+4} \]
- Impulse response \[ h(t) = e^{-2t} - e^{-4t} \]
- Sampled impulse response \[ h(kT) = e^{-2kT} - e^{-4kT}, \quad k = 0, 1, 2, \ldots \]
- Z-transfer function
\[ H(z) = \frac{z}{z - e^{-2T} - z - e^{-4T}} = \frac{(e^{-2T} - e^{-4T})z}{(z - e^{-2T})(z - e^{-4T})} \]

(b) Systems Separated By Samplers
\[ H_1(z) = \frac{z}{z - e^{-2T}}, \quad H_2(z) = \frac{2z}{z - e^{-4T}} \]
\[ H(z) = \frac{2z^2}{(z - e^{-2T})(z - e^{-4T})} \]
- Partial fractions \[ H(z) = \frac{2}{e^{-2T} - e^{-4T}} \left( \frac{e^{-2T}z}{z - e^{-2T}} - \frac{e^{-4T}z}{z - e^{-4T}} \right) \]
- Impulse response sequence
\[ h(kT) = \frac{2}{e^{-2T} - e^{-4T}} \left( e^{-2T} e^{-2kT} - e^{-4T} e^{-4kT} \right) \]
\[ = \frac{2}{e^{-2T} - e^{-4T}} \left( e^{-2(k+1)T} - e^{-4(k+1)T} \right), \quad k = 0, 1, 2, \ldots \]

TF for DAC, Analog Subsystem, ADC Combination
\[ G_{ZA}(s) = G(s)G_{ZOH}(s) = (1 - e^{-sT}) \frac{G(s)}{s} \]
\[ g_{ZA}(t) = g(t) * g_{ZOH}(t) = g_s(t) - g_s(t - T) \]
\[ g_s(t) = \mathcal{L}^{-1} \left\{ \frac{G(s)}{s} \right\} \]
Impulse Response of DAC and Analog Subsystem

(a) Response of analog system to step inputs.
(b) Response of analog system to a unit pulse input.

Sample & z-Transform

- Sampled impulse response
  \[ g_{ZA}(kT) = g_s(kT) - g_s(kT - T) \]
- \( z \)-transfer function of DAC (zero-order hold), analog subsystem, ADC (ideal sampler) cascade
  \[ G_{ZAS}(z) = (1 - z^{-1}) \mathcal{Z}\{g_s^*(t)\} \]
  \[ = (1 - z^{-1}) \mathcal{Z}\{L^{-1} \left(\frac{G(s)}{s}\right)^*\} \]
- Simplify Notation
  \[ G_{ZAS}(z) = (1 - z^{-1}) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \]

Example

Find the \( z \)-domain transfer function for ADC, DAC, and the shown series R-L circuit with the inductor voltage as output.

\[ \begin{align*}
\frac{V_o}{V_{in}} &= \frac{Ls}{Ls+R} = \frac{(L/R)s}{(L/R)s+1} \\
\frac{V_o}{V_{in}} &= \frac{\tau s}{\tau s + 1}, \tau = \frac{L}{R} \\
\text{Use } G_{ZAS}(z) &= (1 - z^{-1}) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\
G_{ZAS}(z) &= (1 - z^{-1}) \mathcal{Z}\left\{\frac{1}{s+1/\tau}\right\} \\
&= \frac{z - 1}{z} \left\{\frac{z}{z - e^{-T/\tau}}\right\} = \frac{z - 1}{z - e^{-T/\tau}}
\end{align*} \]
**Example**

Find the z-domain transfer function of an amplifier, an armature controlled DC motor, ADC and DAC if the s-domain transfer function of the motor and amplifier is

$$G(s) = \frac{\theta(s)}{V_{in}(s)} = \frac{K}{s(\tau_ms + 1)}$$

**Solution**

$$\frac{G(s)}{s} = \frac{K/\tau_m}{s^2(s + 1/\tau_m)}$$

- Partial fraction expansion

$$\frac{G(s)}{s} = K \left[ \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{A_2}{s + 1/\tau_m} \right]$$

$$A_{11} = \frac{1/\tau_m}{s + 1/\tau_m}_{s=0} = 1, A_{12} = \frac{d}{ds} \left( \frac{1/\tau_m}{s + 1/\tau_m} \right)_{s=0} = -1$$

$$A_2 = \frac{1/\tau_m}{s^2}_{s=-1/\tau_m} = \tau_m$$

**Solution (Cont.)**

- Use $G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{ \frac{G(s)}{s} \right\}$

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{ K \left[ \frac{1}{s^2} - \frac{\tau_m}{s} + \frac{\tau_m}{s + 1/\tau_m} \right] \right\}$$

$$= K\tau_m \left[ \frac{T/\tau_m}{z - 1} - 1 + \frac{z - 1}{z - e^{-T/\tau_m}} \right]$$

$$G_{ZAS}(z) = K\tau_m \left[ \frac{T/\tau_m + e^{-T/\tau_m} - 1}{z - 1} + \frac{1 - e^{-T/\tau_m}}{z - e^{-T/\tau_m}} \right]$$

**Systems with Transport Lag**

- Transfer function for systems with a transport delay

$$G(s) = G_a(s)e^{-T_ds}$$

$$T_d = lT - mT, \quad 0 \leq m < 1$$

$l = $ positive integer.

*Example*: time delay $= 3.1$ s, $T = 1$ s $\Rightarrow l = 4$ and $m = 0.9$

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{ e^{-lTz} \mathcal{L}^{-1} \left[ \frac{G_a(s)e^{-(l-m)Ts}}{s} \right] \right\}$$
Delay Theorem

\[ G_{\text{ZAS}}(z) = z^{-l}(1 - z^{-1}) \mathcal{Z}\left\{ \mathcal{L}^{-1}\left[ \frac{G_a(s)}{s} e^{mT_s} \right]^* \right\} \]

- Define the transfer function \( G_s(s) = \frac{G_a(s)}{s} \)
- Advance theorem of Laplace transforms

\[ G_{\text{ZAS}}(z) = z^{-l}(1 - z^{-1}) \mathcal{Z}\left\{ \mathcal{L}^{-1}[G_s(s)e^{mT_s}]^* \right\} = z^{-l}(1 - z^{-1}) \mathcal{Z}\{g_s(kT + mT)\} \]

Use Modified z-Transform

\[ G_{\text{ZAS}}(z) = z^{-(l-1)}(1 - z^{-1})\mathcal{Z}_m\{g_s(kT)\} \]

- Causal time function \( g_s(t) = A_0 + \sum_{i=1}^{n} A_i e^{-p_it} \)
- Sampling & shifting

\[ g_s(kT + mT) = A_0 \mathbf{1}(kT + mT) + \sum_{i=1}^{n} e^{-p_i(kT+mT)} \]

- Shifting a step function has no effect on samples, shifting a decaying exponential function does.

Modified z-Transform

\[ \mathcal{Z}_m\{g_s(kT)\} = \frac{A_0}{z - 1} + \sum_{i=1}^{n} \frac{A_i e^{-p_imT}}{z - e^{-p_iT}} \]

- Transfer function

\[ G_{\text{ZAS}}(z) = z^{-(l-1)}\left( \frac{z - 1}{z} \right) \left[ \frac{A_0}{z - 1} + \sum_{i=1}^{n} \frac{A_i e^{-p_imT}}{z - e^{-p_iT}} \right] = z^{-l}(z - 1) \left[ \frac{A_0}{z - 1} + \sum_{i=1}^{n} \frac{A_i e^{-p_imT}}{z - e^{-p_iT}} \right] \]

Example

If the sampling period is 0.1 s, determine the z-transfer function for the ADC, DAC and analog transfer function

\[ G(s) = \frac{3e^{-0.31s}}{s + 3} \]
Solution

- Delay in terms of the sampling period $T_d = lT - mT$
- Integer $l = [T_d] = [3.1] = 4$
  
  $$m = l - \frac{T_d}{T} = 4 - \frac{0.31}{0.1} = 0.9$$
- Partial fraction expansion $G_s(s) = \frac{3}{s(s+3)} = \frac{1}{s} - \frac{1}{s+3}$
- $z$-transfer function
  
  $$G_{ZAS}(z) = z^{-4}(z - 1) \left\{ \frac{1}{z - 1} - \frac{e^{-0.3\times0.9}}{z - e^{-0.3}} \right\}$$
  
  $$G_{ZAS}(z) = z^{-4} \left\{ \frac{0.2366z + 0.02256}{z - 0.741} \right\}$$

CT Function $g_s(t)$

Block Diagram of Single-loop Digital Control System

Closed-loop Transfer Function

$$G_{cl}(z) = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)}$$

- CL characteristic equation
  
  $$1 + C(z)G_{ZAS}(z) = 0$$
- Closed-loop System Poles
- Roots of the characteristic equation (selected for desired time response specifications as in $s$-domain design)
MATLAB Commands

z-transfer function for ADC, DAC & analog subsystem.

\[
g = \text{tf}(\text{num, den}) \% \text{Enter analog transfer function}
gd = \text{c2d}(g, T, 'method') \% \text{Sampling period } T
\]

'method' = method to obtain the digital transfer function

MATLAB Example

\[
G(s) = \frac{2s^2 + 4s + 3}{s^3 + 4s^2 + 6s + 8}
\]

\[
\begin{align*}
&> g = \text{tf( [2,4,3], [1,4,6,8])} \\
&> gd = \text{c2d}(g, 0.1, 'zoh') \% \text{zero-order hold} \\
&> gd = \text{c2d}(g, 0.1, 'foh') \% \text{first-order hold} \\
&> gd = \text{c2d}(g, 0.1, 'impulse') \% \text{z-transform } \times T
\end{align*}
\]

Transport Delay

\[
\begin{align*}
&> g = \text{tf(3,[1,3],'InputDelay', 0.31)} \\
&\text{Transfer function:} \\
&\quad 3 \\
&\quad \exp(-0.31\times s) \times \frac{\exp(-0.31\times s) \times \frac{\exp(-0.31\times s) \times 3}{s + 3}}{s + 3} \\
&> c2d(g,.1) \\
&\text{Transfer function:} \\
&\quad 0.2366 z + 0.02256 \\
&\quad z^{-4} \times \frac{0.2366 z + 0.02256}{z - 0.7408}
\end{align*}
\]