PD, PI, PID Compensation

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Outline

• PD compensation.
• PI compensation.
• PID compensation.

PD Control

$L =$ loop gain
$s_{cl} =$ desired closed-loop pole location.
• Find a controller $C$ s.t. $\angle CL(s_{cl}) = \pm 180^\circ$ or its odd multiple (angle condition).
• Controller Angle for $s_{cl}$
  $\theta_c = \pm 180^\circ - \angle L(s_{cl})$
• Zero location $a = \frac{\omega_d}{\tan(\theta_c)} + \zeta \omega_n$

Remarks

• Graphical procedures are no longer needed.
• CAD procedure to obtain the design parameters for specified $\zeta$ & $\omega_n$ ($s_{cl}$ known).
  » `evalfr(L,scl)` % Evaluate L at scl
  » `polyval( num,scl)` % Evaluate num at scl
• The complex value and its angle can also be evaluated using any hand calculator.
Procedure: MATLAB or Calculator

1. Calculate $\theta = \pi - \angle L(s_c)$
   
   ```matlab
   theta = pi - angle(evalfr(l, scl))
   ```

2. Calculate zero location using 
   
   $$a = \frac{\omega_d}{\tan(\theta_c)} + \zeta \omega_n$$

3. Calculate new loop gain (with zero)
   
   ```matlab
   Lc = tf([1, a], 1)*L
   ```

4. Calculate gain (magnitude condition)
   
   ```matlab
   K = 1/abs(evalfr(Lc, scl))
   ```

5. Check time response of PD-compensated system. Modify the design to meet the desired specifications if necessary (MATLAB).

Procedure 5.2: Given $e(\infty)$ & $\zeta$

1. Obtain error constant $K_e$ from $e(\infty)$ and determine a system parameter $K_f$ that remains free after $K_e$ is fixed for the system with PD control.

2- Rewrite the closed-loop characteristic equation of the PD controlled system as

$$1 + K_f G_f(s) = 0$$

Procedure 5.2 (Cont.)

3. Obtain $K_f$ corresponding to the desired closed-loop pole location. As in Procedure 5.1, $K_f$ can be obtained by clicking on the MATLAB root locus plot or applying the magnitude condition using MATLAB or a calculator.

4. Calculate the free parameter from the gain $K_f$.

5. Check the time response of the PD compensated system and modify the design to meet the desired specifications if necessary.

Example 5.5

Use a CAD package to design a PD controller for the type I system

$$G(s) = \frac{1}{s(s + 4)}$$

to meet the following specifications

(a) $\zeta = 0.7$ & $\omega_n = 10$ rad/s.

(b) $\zeta = 0.7$ & $e(\infty) = 4\%$ due to a unit ramp.
(a) $\zeta = 0.7$ & $\omega_n = 10 \text{ rad/s}$

**MATLAB**
Calculate pole location & corresponding gain

```matlab
>> scl = 10*exp( j*( pi-acos(0.7) ) )
scl =
   -7.0000 + 7.1414i
>> g=zpk([ ],[0,-4],1);
>> theta=pi-( angle( evalfr( g, scl) ) )
theta =
   1.1731
```

(a) Design (cont.)

```matlab
» a = 10 * sqrt(1-0.7^2)/ tan(theta) + 7
a =
   10.0000
» k =1/abs(evalfr(tf( [1, a],1)*g,scl)) % k =
   10.0000

Gain at scl: $K = 10$
```

**RL of Uncompensated System**

**RL of PD-compensated System**
(b) $\zeta = 0.7, e(\infty) = 4\%$ for unit ramp

$$K_v = \lim_{s \to 0} sC(s)G(s) = \lim_{s \to 0} sK \frac{s + a}{s(s + 4)}$$

$$K_a = \frac{100}{4} = \frac{e(\infty)}{4\%} = 25$$

$$\Rightarrow K_a = 100, a = 100/K$$

• Closed-loop characteristic equation with PD

$$1 + K \frac{s + a}{s(s + 4)} = 0$$

(b) Design (cont.)

$$1 + K \frac{s + a}{s(s + 4)} = 0$$

• Assume that $K$ varies with $K_a$ fixed, then

$$s(s + 4) + Ks + Ka = 0, Ka = 100$$

$$\frac{1}{s} + K \frac{s^2 + 4s + 100}{s^2 + 4s + 100} = 0$$

• RL= circle centered at the origin

**RL of PD-compensated System**

$K \times a$ fixed

**Desired location**: intersection of root locus with $\zeta = 0.7$ radial line.

$K = 10, a = 10$ (same values as in Example 5.3).

MATLAB command to obtain the gain

```matlab
k = 1/abs(evalfr(tf([1,10],[ 1, 4, 0]), scl))
k =
10.0000
```

• Time responses of two designs are identical.
**PI Control**

- Integral control to improve $e(\infty)$ (increases type by one) $\Rightarrow$ worse transient response or instability.
- Add proportional control $\Rightarrow$ controller has a pole and a zero.
- Transfer function of proportional-plus-integral (PI) controller
  \[ C(s) = K_p + \frac{K_i}{s} = K_p \frac{s + a}{s}, \quad a = \frac{K_i}{K_p} \]

**PI Remarks**

- Used in *cascade compensation* (integral term in the feedback path is equivalent to a differentiator in the forward path)
- PI design for a plant transfer function $G(s)$ is the same as PD design of $G(s)/s$.
- A better design is often possible by "almost canceling" the controller zero and the controller pole (negligible effect on time response).

**Procedure 5.3**

1. Design a proportional controller for the system to meet the transient response specifications, i.e. place the dominant closed-loop poles at $s_{cl} = -\zeta \omega_n \pm j \omega_d$.
2. Add a PI controller with the zero location $(-a)$ such that $\phi = 3 - 5^\circ$ (small angle)
   \[ a = \frac{\omega_n}{\zeta + \sqrt{1 - \zeta^2}/\tan(\phi)} \quad \text{or} \quad a = \frac{\zeta \omega_n}{10} \]
3. Tune the gain of the system to moved the closed-loop pole closer to $s_{cl}$.

**Comments**

- Use PI control only if P-control meets the transient response but not the steady-state error specifications. Otherwise, use another control.
- Use pole-zero diagram to prove zero location formula.
Pole-zero Diagram of PI Controller

\[ \tan(180^\circ - \theta_p) = \frac{\omega_d}{\zeta \omega_n} \]

\[ \theta_p = \angle s_{cl} + a \]

\[ \theta_z = \angle(s_{cl} + a) \]

\[ \tan(180^\circ - \theta_z) = \frac{\omega_d}{\zeta \omega_n - a} \]

Proof

- From Figure

\[ \tan(180^\circ - \theta_p) = \frac{\omega_d}{\zeta \omega_n} = \frac{\sqrt{1 - \zeta^2}}{\zeta} \]

\[ \tan(180^\circ - \theta_z) = \frac{\omega_d}{\zeta \omega_n - a} = \frac{\sqrt{1 - \zeta^2}}{\zeta - x} \]

\[ x = a/\omega_n \]

- Controller angle at \( s_{cl} \)

\[ -\phi = \theta_z - \theta_p = (180^\circ - \theta_p) - (180^\circ - \theta_z) \]

Proof (Cont.)

- Trig. Identity \( \tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A) \tan(B)} \)

\[ \tan(\phi) = \frac{\frac{\sqrt{1 - \zeta^2}}{\zeta - x} - \frac{\sqrt{1 - \zeta^2}}{\zeta}}{1 - \frac{\sqrt{1 - \zeta^2}}{\zeta - x} \cdot \frac{\sqrt{1 - \zeta^2}}{\zeta}} = \frac{x\sqrt{1 - \zeta^2}}{1 - \zeta x} \]

- Solve for \( x = \frac{a}{\omega_n} = \frac{\zeta}{10} \)

- Multiplying by \( \omega_n \) gives \( a \)

Controller Angle at \( s_{cl} = \phi (< 3^\circ) vs. \zeta \)

\[ x = \frac{a}{\omega_n} = \frac{\zeta}{10} \]

\[ \phi = \tan^{-1}\left(\frac{x\sqrt{1 - \zeta^2}}{1 - \zeta x}\right) \]

\[ = \tan^{-1}\left(\frac{\zeta\sqrt{1 - \zeta^2}}{10 - \zeta^2}\right) \]

\(< 3^\circ \)
Example 5.6

Design a controller for the position control system to perfectly track a ramp input with a dominant pair with $\zeta = 0.7$ & $\omega_n = 4 \text{ rad/s}$.

$$G(s) = \frac{1}{s(s + 10)}$$

Solution

- To use the PD procedure for PI design, consider
  $$G_i(s) = \frac{G(s)}{s} = \frac{1}{s^2(s + 10)}$$
- Procedure 5.1 with modified transfer function.
- Unstable for all gains (see root locus plot).
- Controller must provide an angle $\approx -249^\circ$ at the desired closed-loop pole location.
- Zero at $-1.732$.
- Cursor at desired pole location on compensated system RL gives a gain of about 40.6.
Analytical Design

- Closed-loop characteristic polynomial
  \[ s^3 + 10s^2 + Ks + Kz = (s + \alpha)(s^2 + 2\zeta \omega_n s + \omega_n^2) \]
  \[ = s^3 + (\alpha + 2\zeta \omega_n)s^2 + (2\zeta \omega_n \alpha + \omega_n^2)s + \alpha \omega_n^2 \]
  \[ \alpha = 10 - 2\zeta \omega_n = 10 - 2(0.7)(4) = 4.4 \]
  \[ K = \omega_n(\alpha + 2\zeta \omega_n) = 4 \times [2(0.7)(4.4) + 4] = 40.64 \]
  \[ a = \frac{\alpha \omega_n^2}{K} = \frac{4.4 \times 4^2}{40.64} = 1.732 \]
- Values obtained earlier, approximately.

MATLAB

```
scl = 4*exp(j*(pi - acos(0.7)))
scl =
  -2.8000 + 2.8566i
theta = pi + angle(polyval([1, 10, 0, 0], scl))
theta =
  1.9285
a = 4*sqrt(1-.7^2)/tan(theta)+ 4*.7
a =
  1.7323
k = abs(polyval([1, 10, 0, 0], scl )/ polyval([1,a], scl )
k =
  40.6400
```

Design I Results

Closed-loop transfer function for Design I

\[ G_{cl}(s) = \frac{40.64}{s + 1.732} \frac{s + 1.732}{s^3 + 10s^2 + 40.64s + 69.28} \]
\[ = 40.64 \frac{s + 4.4}{(s + 4.4)(s^2 + 5.6s + 16)} \]
- Zero close to the closed-loop poles ⇒ excessive PO
  (see step response together with the response for a later design: Design II).

Step Response of PI-compensated System
Design II: Procedure 5.3

For $\zeta = 0.7, K \approx 51.02$ & $\omega_n = 7.143$ rad/s.

Zero location

\[ a = \frac{\omega_n}{\zeta + \sqrt{1 - \zeta^2}/\tan(\phi)} = \frac{7.143}{0.7 + \sqrt{1 - 0.49}/\tan(3^\circ)} \approx 0.5 \]

Closed-loop transfer function

\[ G_{cl}(s) = \frac{50(s + 0.5)}{s^3 + 10s^2 + 50s + 25} \]

\[ = \frac{50(s + 0.5)}{(s + 0.559)(s^2 + 9.441s + 44.7225)} \]

Gain slightly reduced to improve response.

Compare Designs I & II

- Design II dominant poles:
  \[ \zeta = 0.7, \omega_n = 6.6875 \text{ rad/s} \]

- Time response for Designs I and II
- PO for Design II (almost pole-zero cancellation) \ll PO for Design I (II better)

PID (proportional-plus-integral-plus-derivative) Control

- Improves both transient & steady-state response.
- $K_p, K_i, K_d =$ proportional, integral, derivative gain, resp.
- Controller zeros: real or complex conjugate.

\[ C(s) = K_p + \frac{K_i}{s} + K_d s = K_d s^2 + 2\zeta \omega_n s + \omega_n^2 \]

\[ 2\zeta \omega_n = \frac{K_p}{K_d}, \omega_n^2 = \frac{K_i}{K_d} \]

PID Design Procedures

1. Cancel real or complex conjugate LHP poles.
2. Cancel pole closest to (not on) the imaginary axis then add a PD controller by applying Procedures 5.1 or 5.2 to the reduced transfer function with an added pole at the origin.
3. Follow Procedure 5.3 with the proportional control design step modified to PD design. The PD design meets the transient response specifications and PI control is then added to improve the steady-state response.
Example 5.7

Design a PID controller for the transfer function to obtain zero $e(\infty)$ due to ramp, $\zeta = 0.7$ and $\omega_n = 4$ rad/s.

$$G(s) = \frac{1}{s(s + 1)(s + 10)}$$

Design I

- Cancel pole at $-1$ with a zero & add an integrator
  $$G_i(s) = \frac{1}{s^2(s + 10)}$$
- Same as transfer function of Example 5.6
- Add PD control of Example 5.6
- PID controller
  $$C(s) = 50 \frac{(s + 1)(s + 0.5)}{s}$$

Design II

- Design PD control to meet transient response specs.
- Select $\omega_n = 5$ rad/s (anticipate effect of adding PI).

Design II (Cont.)

- Obtain PI zero (approx. same as $\zeta \omega_n / 10$)
  $$b = \frac{5}{0.7 + \sqrt{1-0.49}/\tan(3\pi/180)}$$

Transfer function for Design II (reduce gain to 40 to correct for PI)

$$C(s) = 40 \frac{(s + 0.349)(s + 2.326)}{s}$$
Conclusions

• Compare step responses for Designs I and II: Design I is superior because the zeros in Design II result in excessive overshoot.

• Plant transfer function favors pole cancellation in this case because the remaining real axis pole is far in the LHP. If the remaining pole is at \(-3\), say, the second design procedure would give better results.

• MORAL: No easy solutions in design, only recipes to experiment with until satisfactory results are obtained.

Example 5.8

Design a PID controller for the transfer function to obtain zero \(e(\infty)\) due to step,

\[ \zeta = 0.7 \text{ and } \omega_n \geq 4 \text{ rad/s} \]

\[ G(s) = \frac{1}{(s+10)(s^2+2s+10)} \]

Solution

• Complex conjugate pair slows down the time response.
• Third pole is far in the LHP.
• Cancel the complex conjugate poles with zeros and add an integrator

\[ G(s) = \frac{1}{s(s+10)} \]
Solution (cont.)

- Stable for all gains.
- Closed-loop characteristic polynomial
  \[ s^2 + 10s + K = s^2 + 2\zeta \omega_n s + \omega_n^2 \]
- Equate coefficients with \( \zeta = 0.7 \)
  \[ \omega_n = \frac{10}{2\zeta} = \frac{5}{0.7} = 7.143 \]
- (meets design specs.)
- PID controller
  \[ G(s) = 51.02 \frac{s^2 + 2s + 10}{s} \]

Step Response of PID-compensated System

In practice, pole-zero cancellation may not occur but near cancellation is sufficient to obtain a satisfactory time response.