Root Locus Design

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Outline

• Pole locations and time response.
• Root locus compensation.
• Proportional control.
• PD control.
• Effect of configuration on feedback control.

Pole Locations & Time Responses

1. Real pole: exponential time response which decays for LHP poles and increases for RHP poles.
2. Pole at origin: a unit step.
3. Complex conjugate poles: oscillatory response which decays exponentially for LHP poles and increases exponentially for RHP poles.
4. Real part of pole determines the rate of exponential change.
5. Imaginary part of pole determines the frequency of oscillations.
6. Imaginary axis poles are associated with sustained oscillations.

s-domain Poles & Time Response

1. Real pole: exponential time response which decays for LHP poles and increases for RHP poles.
2. Pole at origin: a unit step.
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4. Real part of pole determines the rate of exponential change.
5. Imaginary part of pole determines the frequency of oscillations.
6. Imaginary axis poles are associated with sustained oscillations.
Objective

Select a desirable time response through the selection of closed-loop pole locations.
1. Try proportional control.
2. If this fails, add a dynamic controller (poles & zeros).
3. Add a zero to the system to improve its time response.
4. Add a pole at the origin to reduce steady-state error (may adversely affect the transient response).
5. Add two zeros & a pole at the origin to improve both transient and steady-state performance.

Common Control Configurations

(a) Cascade controller.
(b) Feedback controller.
(c) Inner loop feedback controller.

Cascade & Feedback Controller

Inner Loop Feedback Controller
**Proportional Control Example**

Position control system: armature controlled DC motor, power amplifier, gear train.
- Output = load angular position
- Input = motor armature voltage
- Transfer function
  \[ G(s) = \frac{K}{s(s + p)} \]
- Design a proportional controller for
  a) a given damping ratio \( \zeta \).
  b) a given undamped natural frequency \( \omega_n \).

**Solution**

- Root locus remains in the LHP.
- Closed-loop characteristic equation:
  \[ s(s + p) + K = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]
  \[ \omega_n = \sqrt{K}, \quad \zeta = \frac{p}{2\sqrt{K}} \]
  a) \( \zeta \) & \( p \) given:
    \[ K = \left( \frac{p}{2\zeta} \right)^2, \quad \omega_n = \frac{p}{2\zeta} \]
  a) \( \omega_n \) & \( p \) given:
    \[ K = \omega_n^2, \quad \zeta = \frac{p}{2\omega_n} \]

**PD Control**

- Derivative is only approximately realizable.
- Differentiating a noisy input gives large errors.
- Measure derivative of output to obtain a more practical PD control.
  \[ C(s) = K_p + K_ds = K_d(s + a) \]
  \[ a = \frac{K_p}{K_d} \]

**Effect of Configuration on PD**

Cascade controller

\[ G_{cl}(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{K_d(s + a)N(s)}{D(s) + K_d(s + a)N(s)} \]

\( N(s), D(s) = \) numerator & denominator of open-loop gain.
- Pole cancels zero if the loop gain has a pole at \(-a\).
- Otherwise, closed-loop system has a zero at \(-a\).
  (alters time response \( \uparrow \) PO)
Block Diagram of PD Feedback Compensated System

\[
R(s) \xrightarrow{\text{Preamplifier } K_p} + \xrightarrow{\text{Amplifier } K_a} \xrightarrow{\text{Plant } G(s)} Y(s)
\]

\[K_p = \text{preamplifier in cascade with loop},\]
\[K_a = \text{amplifier in the forward path}\]
\[G_{cl}(s) = \frac{K_pK_aG(s)}{1 + K_aG(s)C(s)} = \frac{K_pK_aN(s)}{D(s) + K_aK_d(s + a)N(s)}\]

• No closed-loop zero at \(-a\) for feedback PD.

Feedback PD Remarks

\[
G_{cl}(s) = \frac{K_pK_aG(s)}{1 + K_aG(s)C(s)} = \frac{K_pK_aN(s)}{D(s) + K_aK_d(s + a)N(s)}
\]

• Loop gain is the same for cascade & feedback PD.
• If \(D(s) = (s + a)D_1(s)\) has a pole at \(-a\): feedback-compensated system has a closed-loop pole at \(-a\) that cancels in the loop gain i.e. does not appear in the root locus.

Free Design Parameters

• Feedback & cascade compensation: 2 free design parameters
  \(\Rightarrow\) 2 design criteria can be selected.

Example

\[G(s) = \frac{K}{s(s + p)}\]

Design a PD controller for the type 1 system to obtain

a) Specified \(\zeta\) & \(\omega_n\).

b) Specified \(\zeta\) & steady-state error \(e(\infty)\)% due to a ramp input.

Consider both cascade and feedback compensation and compare them using a numerical example.
Solution
• Root locus for PD (stable for all gains $K > 0$)
• Root locus: locus of pole locations for $K > 0$.
• Obtain PD control closed-loop characteristic equation and equate coefficients.

Equating Coefficients

$$G(s) = \frac{1}{s(s + p)}, \quad C(s) = K(s + a)$$

$K = K_d$ for cascade control
$K = K_a K_d$ for FB control.

$$s^2 + ps + K(s + a) = s^2 + (p + K)s + Ka$$
$$= s^2 + 2\zeta\omega_n s + \omega_n^2$$

Equate coefficients (both configurations)

$$Ka = \omega_n^2$$
$$p + K = 2\zeta\omega_n$$

a) Specified $\zeta$ & $\omega_n$

• Same equations for $K$ and $a$ in cascade & feedback control.
• Feedback control requires amplifier gain settings that yield zero steady-state error due to unit step (see part b).

$$Ka = \omega_n^2$$
$$p + K = 2\zeta\omega_n$$

• Solve for $K$ and $a$

$$K = 2\zeta\omega_n - p, \quad a = \frac{\omega_n^2}{2\zeta\omega_n - p}$$

Cascade Control

a) $p = 4$

• Specifications: $\zeta = 0.7, \quad \omega_n = 10\frac{\text{rad}}{s}$

$$K = K_d = 2\zeta\omega_n - p = 14 - 4 = 10$$

$$a = \frac{\omega_n^2}{K} = \frac{100}{10} = 10$$
Feedback Control

a) \( p = 4 \),

- Specifications: \( \zeta = 0.7, \ \omega_n = 10 \frac{\text{rad}}{s} \)
  \[ K = K_a K_d = 2\zeta \omega_n - p = 10 \]
  \[ a = \frac{\omega_n}{K} = 10 \]
- Steady-state error due to step: \( 1 - G_{cl}(0) = 0 \)
  \[ G_{cl}(s) = \frac{K_p K_a}{s(s + 4) + 10(s + 10)} \]
- Select amplifier gains so that the numerator = 100 for unity steady state-output due to unit step.
  e.g. \( K_p = 10, K_a = 10, K_d = 1 \) and \( a = 10 \).

Cascade Versus Feedback Control

a) \( p = 4, \ \zeta = 0.7, \ \omega_n = 10 \frac{\text{rad}}{s} \)

Cascade Control:
Higher PO (>10%) with a faster response due to the zero (1.5 pole distance from the \( j\omega \)-axis).
Feedback Control:
PO \( \approx 4.6\% \)
Predicted exactly using the 2\textsuperscript{nd} order formula

b) Specified \( \zeta \) & \( e(\infty)\% \) due to a ramp input (Cascade)

\[ Ka = \omega_n^2, \quad p + K = 2\zeta \omega_n \]
\[ L(s) = \frac{K_d(s + a)}{s(s + p)} \Rightarrow K_v = \frac{Ka}{p} = \frac{100}{e(\infty)\%} \]
\[ \omega_n = \sqrt{Ka} = \sqrt{pK_v} \]
Cannot freely select \( \omega_n \) for specified \( e(\infty) \)
\[ K = 2\zeta \omega_n - p = 2\zeta \sqrt{pK_v} - p \]
\[ a = \frac{\omega_n^2}{K} = \frac{pK_v}{2\zeta \sqrt{pK_v} - p} \]
Cascade Compensation

b) For $p = 4$,

- Specifications: $\zeta = 0.7$, steady-state error $= 4\%$
  
  $K_v = \frac{Ka}{p} = \frac{100}{e(\infty)\%} = 25$

  $Ka = 100$

  $\omega_n = \sqrt{Ka} = \sqrt{pK_v} = 10 \text{ rad/s}$

  $K = 2\zeta\omega_n - p = 14 - 4 = 10$

- Requires $K = 10, a = 10$, same values as (a)

b) Specified $\zeta$ & $e(\infty)$% ramp input (Feedback)

$$G_{cl}(s) = \frac{K_pK_aN(s)}{D(s) + K_aK_d(s + a)N(s)} = \frac{K_pK_a}{s(s + p) + K_aK_d(s + a)}$$

$$R(s) - Y(s) = R(s) \left[ 1 - \frac{K_pK_a}{s^2 + (p + K)s + Ka} \right]$$

$$= R(s) \left[ \frac{s^2 + (p + K)s + Ka - K_pK_a}{s^2 + (p + K)s + Ka} \right]$$

$$Ka = K_pK_a$$

$$e(\infty) = \lim_{s \to 0} \frac{\text{Output}}{\text{Input}} = \frac{s}{s^2} \left[ \frac{s^2 + (p + K)s + Ka - K_pK_a}{s^2 + (p + K)s + Ka} \right] = \frac{p + K}{Ka}$$

Design (Specified $\zeta$ & $e(\infty)$% )

$$Ka = \omega_n^2, \quad p + K = 2\zeta\omega_n$$

$$Ka = \frac{p + K}{e(\infty)} = \frac{2\zeta\omega_n}{e(\infty)} = \omega_n^2$$

- Calculate
  
  $$\omega_n = \frac{2\zeta}{e(\infty)}$$

- Solve for $K$ and $a$
  
  $$K = \frac{4\zeta^2}{e(\infty)} - p, \quad a = \frac{\omega_n^2}{K} = \frac{4\zeta^2}{e(\infty)(4\zeta^2 - pe(\infty))}$$

Steady-state Error (Feedback)

- Unlike cascade control, can specify $\omega_n$ and $e(\infty)$ with $\zeta$ free.

  $$Ka = \omega_n^2, \quad p + K = 2\zeta\omega_n$$

  $$Ka = \frac{p + K}{e(\infty)} = \frac{2\zeta\omega_n}{e(\infty)}$$

- Calculate $\zeta = \frac{e(\infty)\omega_n}{2}$

- Solve for $K$ and $a$

  $$K = \frac{4\zeta^2}{e(\infty)} - p, \quad a = \frac{\omega_n^2}{K} = \frac{4\zeta^2}{e(\infty)(4\zeta^2 - pe(\infty))}$$
Feedback Compensation

\[ \omega_n = \frac{2\zeta}{e(\infty)} = \frac{1.4}{0.04} = 35 \frac{rad}{s} . \]

\[ K = K_d K_a = 2\zeta \omega_n - p = 45 \]

\[ a = \frac{\omega_n^2}{K} = 27.222, K_p K_a = K_a \]

- Closed-loop transfer function

\[ \frac{K_p K_a}{s^2 + (p + K)s + Ka} = \frac{1225}{s^2 + 449s + 1225} \]

Step response: PD cascade & PD feedback controller, given \( \zeta \) and steady-state error.

Comparison (b)

- Cascade Compensation \( K = 10, a = 10 \), same values as (a)
- \( \omega_n = 10 \), high PO, zero close to complex conjugate poles.
- Feedback Compensation \( K = K_d K_a = 45, a = 27.222, K_p K_a = 1225 \)
- Closed-loop transfer function

\[ \frac{K_p K_a}{s^2 + (p + K)s + Ka} = \frac{1225}{s^2 + 449s + 1225} \]

- PO for the feedback compensated case is still 4.6%.
  1. Superior
  2. Requires high gain amplifiers (may cause nonlinear behavior such as saturation).