Stability of Digital Control Systems

M. Sami Fadali
Professor of Electrical Engineering
UNR

Outline

- Asymptotic stability.
- Input-output stability.
- Internal stability.
- Stability conditions.
- Routh-Hurwitz criterion.
- Jury test.

Asymptotic Stability

Response due to **any** initial conditions decays to zero asymptotically in the steady state

$$\mathcal{L}im_{k\to\infty} y(k) = 0$$

Marginal Stability: response due to any initial conditions remains bounded but does not decay to zero.

Bounded-Input-Bounded-Output (BIBO) Stability

The response due to any bounded input

$$|u(k)| < b_u, \qquad k = 0,1,2,\ldots$$

$$0 < b_u < \infty$$

remains bounded.

$$|y(k)| < b_y, \qquad k = 0,1,2,\ldots$$

$$0 < b_y < \infty$$

Stable Z-Domain Pole Locations

Sampled exponential and its z-transform with p real or complex

$$p^k, k = 0, 1, 2, \ldots \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-p}$$

$$\mathcal{L}im_{k\to\infty} |p|^k = \begin{cases} 0, & |p| < 1 \\ 1, & |p| = 1 \\ \infty, & |p| > 1 \end{cases}$$

Time Sequence

$$f(k) = \sum_{i=1}^{n} A_i p_i^k, k = 0, 1, 2, \ldots \overset{\mathcal{Z}}{\longleftrightarrow} F(z) = \sum_{i=1}^{n} \frac{A_i z}{z - p_i}$$

- Bounded sequence for poles in the closed unit disc (i.e. on or inside the unit circle).
- Sequence decays exponentially for poles in the open unit disc (i.e. inside the unit circle).
- Unbounded sequence for repeated poles on the unit circle.
- For real time sequences poles and partial fraction coefficients are either real or complex conjugate pairs.

Theorem: Asymptotic Stability

Asymptotic Stability: response to any ICs decays to zero asymptotically.

Thm. In the absence of pole-zero cancellation, a LTI digital system is (i) asymptotically stable if its transfer function poles are in the open unit disc and (ii) marginally stable if the poles are in the closed unit disc with no repeated poles on the unit circle.

LTI Model:

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= b_m u(k+n) + b_{m-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)$$
$$k = 0,1,2, \ldots$$

Proof

- Response of the system due to ICs $y(0), y(1), \ldots, y(n)$

$$Y(z) = \frac{N(z)}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}$$

1. Assume no pole-zero cancellation.
2. $Y(z)$ denominator= z-transfer function denominator.
3. Poles of $Y(z)$ = poles of transfer function.
4. $Y(z)$ due to ICs: bounded for system poles in the closed unit disc unit (no repeated poles on the unit circle) & decays exponentially for system poles in the open unit disc (inside the unit circle).
5. No cancellation so all poles are considered.

Example

Determine the asymptotic stability of the following systems:

$$a)\ H(z) = \frac{4(z-2)}{(z-2)(z-0.1)},\ b)\ H(z) = \frac{4(z-0.2)}{(z-0.2)(z-0.1)}$$

$$c)\ H(z) = \frac{5(z-0.3)}{(z-0.2)(z-0.1)},\ d)\ H(z) = \frac{8(z-0.2)}{(z-0.1)(z-1)}$$

- Use Theorem 1 (a) and (b) **without** pole-zero cancellation.
- Ignore zeros, (do not affect response due to ICs)

Solution (a, b)

$$a)\ H(z) = \frac{4(z-2)}{(z-2)(z-0.1)}$$

- Pole outside the unit circle $\Rightarrow$ unstable

$$b)\ H(z) = \frac{4(z-0.2)}{(z-0.2)(z-0.1)}$$

- All poles inside the unit circle $\Rightarrow$ asymptotically stable.

Solution (c, d)

$$c)\ H(z) = \frac{5(z-0.3)}{(z-0.2)(z-0.1)}$$

- All poles inside the unit circle $\Rightarrow$ asymptotically stable.

$$d)\ H(z) = \frac{8(z-0.2)}{(z-0.1)(z-1)}$$

- One pole on the unit circle $\Rightarrow$ marginally stable: not asymptotically stable.

BIBO Stability

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i), \qquad k = 0, 1, 2, \ldots$$

Is system BIBO stable if its impulse response $h(k)$ is bounded? NO

Counterexample: bounded & strictly positive impulse response $0 < b_{h1} < h(k) < b_{h2} < \infty,\ k = 0, 1, 2, \ldots$

Bounded input: $u(i) = 1, i = 0, 1, 2, \ldots$

Unbounded output:

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i) > b_{h1} \sum_{i=0}^{k} 1, \qquad k = 0, 1, 2, \ldots$$

Theorem 4.2:BIBO Stability

A discrete-time linear system is BIBO stable if and only if its impulse response sequence is absolutely summable i.e.

$$\sum_{i=0}^{\infty} |h(i)| < \infty$$

Proof of Necessity (Only if)

$$y(k) = \sum_{i=0}^{k} h(i)u(k-i), \qquad k = 0, 1, 2, \ldots$$

- Assume the system is BIBO stable but the impulse response is not absolutely summable.
- Input $u(k-i) = \begin{cases} 1, & h(i) \geq 0 \\ -1, & h(i) < 0 \end{cases}$

$$y(k) = \sum_{i=0}^{k} |h(i)|, \qquad y(k) \to \infty \; as \; k \to \infty$$

- Contradiction: unbounded output with a bounded input.

Proof of Sufficiency (If)

- Assume an absolutely summable impulse response and show that the system is BIBO stable.
- Use the input bound b_u in the convolution summation $|u(k)| < b_u$

$$|y(k)| \leq \sum_{i=0}^{k} |h(i)||u(k-i)|$$

$$< b_u \sum_{i=0}^{k} |h(i)| < \infty, \qquad \forall k \geq 0$$

Theorem 4.3: BIBO Stability

A discrete-time linear system is BIBO stable if and only if the poles of its transfer function lie inside the unit circle.

Proof of Necessity

$$h(k) = \sum_{i=1}^{n} A_i p_i^k \,, k = 0, 1, 2, \ldots \overset{\mathcal{Z}}{\longleftrightarrow} H(z) = \sum_{i=1}^{n} \frac{A_i\, z}{z - p_i}$$

- Impulse response is bounded if the poles of the transfer function are in the closed unit disc and decays exponentially if the poles are in the open unit disc.
- Systems with a bounded impulse response that does not decay exponentially are not BIBO stable.

Proof of Sufficiency

- Assume exponentially decaying impulse response (i.e. poles inside the unit circle).

  A_r is the coefficient of largest magnitude $|p_s| < 1$ is the largest pole magnitude.
- The impulse response is bounded by

$$|h(k)| = \left|\sum_{i=1}^{n} A_i p_i^k\right| \le \sum_{i=1}^{n} |A_i||p_i|^k \le n|A_r||p_s|^k,$$

$$k = 0,1,2, \ldots$$

$$\sum_{k=0}^{\infty} |h(k)| \le n|A_r| \sum_{k=0}^{\infty} |p_s|^k = \frac{n|A_r|}{1 - |p_s|} < \infty$$

Example 4.2

Investigate the BIBO stability of systems with the impulse response

$$h(k) = \begin{cases} K, & 0 \le k \le m < \infty \\ 0, & elsewhere \end{cases}$$

where K is a finite constant.

Solution

BIBO stable since the impulse response satisfies

$$\sum_{k=0}^{\infty} |h(k)| = \sum_{k=0}^{m} |h(k)| = (m+1)K < \infty$$

- Let K = upper bound for any impulse response of finite duration.
- Any FIR system is BIBO stable.

Example 4.3

Investigate BIBO stability for Example 4.1

$$a)\ H(z) = \frac{4(z-2)}{(z-2)(z-0.1)},\ b)\ H(z) = \frac{4(z-0.2)}{(z-0.2)(z-0.1)}$$

$$c)\ H(z) = \frac{5(z-0.3)}{(z-0.2)(z-0.1)},\ d)\ H(z) = \frac{8(z-0.2)}{(z-0.1)(z-1)}$$

- Use Theorem 4.3 with pole-zero cancellation.

Solution (a, b, c)

$$a)\ H(z) = \frac{4(z-2)}{(z-2)(z-0.1)},\ b)\ H(z) = \frac{4(z-0.2)}{(z-0.2)(z-0.1)}$$

$$c)\ H(z) = \frac{5(z-0.3)}{(z-0.2)(z-0.1)}$$

After pole-zero cancellation

a) BIBO stable, all poles inside unit circle.

b) BIBO stable, all poles inside unit circle.

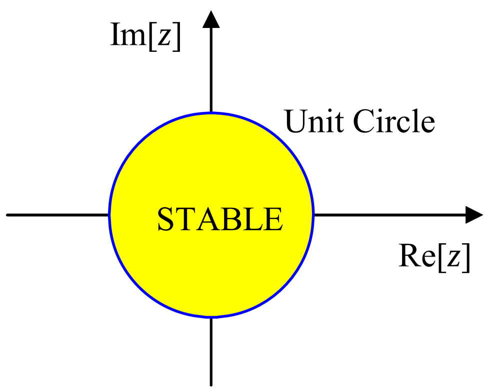
c) BIBO stable, all poles inside unit circle.

Solution (d)

$$d)\ H(z) = \frac{8(z-0.2)}{(z-0.1)(z-1)}$$

- Not BIBO stable, a pole on unit circle.

Z-plane Stable Pole Locations



MATLAB Stability Determination

Obtain roots of polynomial:

» **roots(den) % denominator coeffts. den**

» **zpk(g) %g** = transfer function

Stable for roots inside the unit circle.

MATLAB: ddamp

Pole	Magnitude	Damping	Frequency (rad/TimeUnit)	Time Constant (TimeUnit)
-5.00e-02 + 1.32e-01i	1.41e-01	7.11e-01	2.75e+00	5.11e-01
-5.00e-02 - 1.32e-01i	1.41e-01	7.11e-01	2.75e+00	5.11e-01

>> **ddamp([1,.1,.02])**

Gives the pole locations, ζ and ω_n

- Closed-loop transfer function

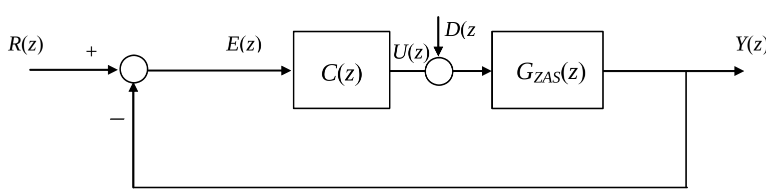
>> **H = feedback(gforward, gfeedback, ±1)**

- **Default:** negative feedback

Internal Stability

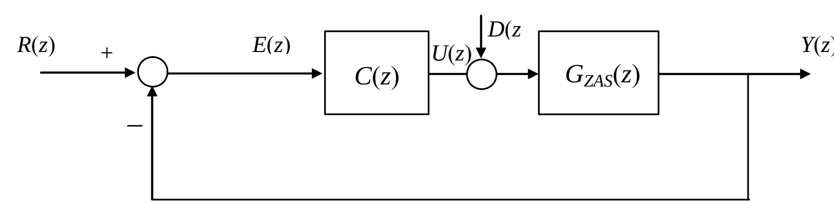
If all the transfer functions that relate the system inputs (R and D) to the possible system outputs (Y and U) are BIBO stable, then the system is internally stable.

$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} \dfrac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} & \dfrac{G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \\ \dfrac{C(z)}{1 + C(z)G_{ZAS}(z)} & \dfrac{-C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \end{bmatrix} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$



Theorem 4-4

Assume no unstable pole-zero cancellation. The system of Figure 4-3 is internally stable if and only if all the closed-loop poles are in the open unit disc.

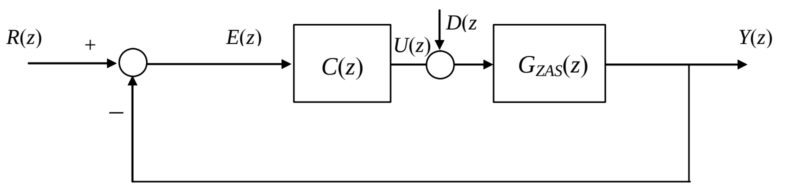


Transfer Functions

$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \frac{\begin{bmatrix} G_{ZAS}(z)C(z) & G_{ZAS}(z) \\ C(z) & -C(z)G_{ZAS}(z) \end{bmatrix}}{1 + G_{ZAS}(z)C(z)} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$

- Coprime polynomials (no common factors)

$$C(z) = \frac{N_C(z)}{D_C(z)}, \qquad G_{ZAS}(z) = \frac{N_G(z)}{D_G(z)}$$



Proof (Necessity: only if)

- Substitute $C(z) = \frac{N_C(z)}{D_C(z)}$, $G_{ZAS}(z) = \frac{N_G(z)}{D_G(z)}$

$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \frac{\begin{bmatrix} N_c N_G & D_c N_G \\ N_c D_G & -N_c N_G \end{bmatrix}}{D_c D_G + N_c N_G} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$

- Internally stable (all transfer functions are asymptotically stable) + coprime

⇒ characteristic polynomial $D_c D_G + N_c N_G$ has no zeros on or outside the unit circle.

Proof (Sufficiency: if)

- Substitute $C(z) = \frac{N_C(z)}{D_C(z)}$, $G_{ZAS}(z) = \frac{N_G(z)}{D_G(z)}$

$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \frac{\begin{bmatrix} N_c N_G & D_c N_G \\ N_c D_G & -N_c N_G \end{bmatrix}}{D_c D_G + N_c N_G} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$

- Characteristic polynomial $D_c D_G + N_c N_G$
- No zeros on or outside the unit circle

⇒ the systems with the 4 TFs are asymptotically stable ⇒ closed-loop system is internally stable.

Theorem 4-5

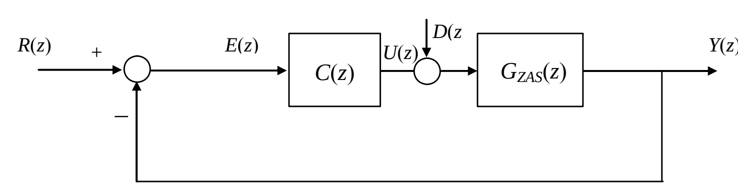
The system is internally stable if and only if:

1. The characteristic polynomial

$$1 + C(z)G_{ZAS}(z)$$

   has no zeros on or outside the unit circle.

2. The loop gain $C(z)G_{ZAS}(z)$ has no pole-zero cancellation on or outside the unit circle .



Proof: Necessity

(i) $1 + C(z)G_{ZAS}(z)$: no zeros on or outside the unit circle for internal stability (Theorem 4.4)

(ii) $|p| > 1$ cancels ⇒ unstable

$$C(z)G_{ZAS}(z) = \frac{N_C(z)N_G(z)}{D_C(z)D_G(z)} = \frac{(z-p)N_1(z)}{(z-p)D_1(z)}$$

$$1 + C(z)G_{ZAS}(z) = \frac{D_C(z)D_G(z) + N_C(z)N_G(z)}{D_C(z)D_G(z)}$$

$$= \frac{(z-p)[N_1(z) + D_1(z)]}{(z-p)D_1(z)} = \frac{\text{c. l. charact. poly.}}{\text{o. l. charact. poly.}}$$

p is both an open-loop and a closed-loop pole

Proof: Sufficiency

$$1 + C(z)G_{ZAS}(z) = \frac{D_C(z)D_G(z) + N_C(z)N_G(z)}{D_C(z)D_G(z)}$$

i. No zeros (c.l. poles) on or outside the unit circle
ii. No unstable pole-zero cancellation for the closed-loop system if there is no unstable pole-zero cancellation for

$$C(z)G_{ZAS}(z) = \frac{N_C(z)N_G(z)}{D_C(z)D_G(z)}$$

Sufficient for internal stability (Theorem 4.4)

Example 4-4

Transfer function of isothermal chemical reactor

$$G(s) = \frac{0.5848(-0.3549s + 1)}{0.1828s^2 + 0.8627s + 1}$$

- Determine $G_{ZAS}(z)$ with $T = 0.1$
- Verify that the resulting feedback system is not internally stable with the feedback controller

$$C(z) = -\frac{10(z - 0.8149)(z - 0.7655)}{(z-1)(z - 1.334)}$$

Solution

- Discretized process transfer function

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= \frac{-0.075997(z - 1.334)}{(z - 0.8149)(z - 0.7655)}$$

- Transfer function from reference input to output

$$\frac{Y(z)}{R(z)} = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} = \frac{0.75997}{z - 0.24}$$

- All its poles are inside the unit circle (some cancel)

Solution: Control Variable

$$\frac{U(z)}{R(z)} = \frac{C(z)}{1 + C(z)G_{ZAS}(z)}$$
$$= \frac{-10(z - 0.8149)(z - 0.7655)}{(z - 0.24)(z - 1.334)}$$

- Pole at 1.334 outside the unit circle.
- The control variable is unbounded even when the reference input is bounded.

Solution: Theorem 4-5

$$C(z)G_{ZAS}(z) = \frac{-10(z - 0.8149)(z - 0.7655)}{(z - 0.24)(z - 1.334)} \times \frac{-0.075997(z - 1.334)}{(z - 0.8149)(z - 0.7655)}$$

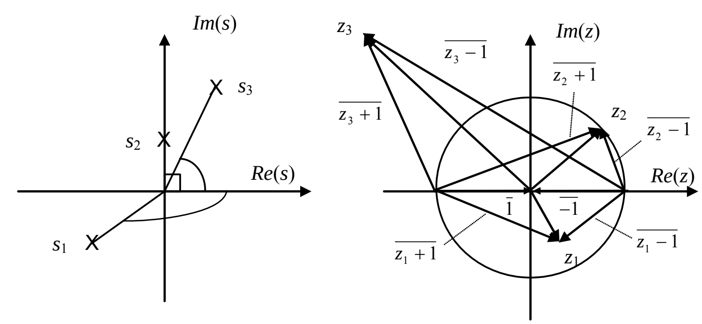
- Violates condition (ii) of Theorem 4-5: Unstable pole at 1.334 cancels in the loop gain.

Routh-Hurwitz Criterion

1. Transform the inside of the unit circle to the LHP (*bilinear* transformation).
2. Use the Routh-Hurwitz criterion for the investigation of discrete-time system stability.

$$z = \frac{1 + w}{1 - w} \Leftrightarrow w = \frac{z - 1}{z + 1}$$

Geometric Interpretation



$$\angle s = \angle(z-1) - \angle(z+1)$$

$$\angle s \begin{cases} < 90^\circ \rightarrow \text{inside unit circle} \\ > 90^\circ \rightarrow \text{outside unit circle} \\ = \pm 90^\circ \rightarrow \text{on unit circle} \end{cases}$$

Advantages/Disadvantages

- Easy stability test for low-order polynomials.
- Difficult for high order z-polynomials.
- For high order polynomials, use symbolic manipulation.

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$$\xrightarrow{z=\frac{1+w}{1-w}}$$

$$a_n\left(\frac{1+w}{1-w}\right)^n + a_{n-1}\left(\frac{1+w}{1-w}\right)^{n-1} + \cdots + a_1\left(\frac{1+w}{1-w}\right) + a_0$$

Example 4.4

Find stability conditions for

a) The first order polynomial

$$a_1 z + a_0, \qquad a_1 > 0$$

b) The second order polynomial

$$a_2 z^2 + a_1 z + a_0, \qquad a_2 > 0$$

Solution: 1st order

- Solve for the root.
- Stability conditions

$$\left|\frac{a_0}{a_1}\right| < 1$$

Solution: 2nd order

$$z_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$$

- Stability determination by solving for roots is difficult.
- Monic polynomial
  - constant term = product of poles
- For pole magnitudes < 1
  - Necessary stability condition
  - Sufficient for complex conjugate poles

$$\left|\frac{a_0}{a_2}\right| < 1 \Leftrightarrow -a_0 < a_2 \;\&\; a_0 < a_2$$

Bilinear Transformation

$$a_2\left(\frac{1+w}{1-w}\right)^2 + a_1\left(\frac{1+w}{1-w}\right) + a_0$$

$$(a_2 - a_1 + a_0)w^2 + 2(a_2 - a_0)w + (a_2 + a_1 + a_0)$$

- Routh-Hurwitz criterion: poles of 2nd order w-polynomial remain in the LHP iff its coefficients are all positive.

$$a_2 - a_1 + a_0 > 0, a_2 - a_0 > 0, a_2 + a_1 + a_0 > 0$$

- Adding the first and third conditions gives the condition obtained earlier $a_2 + a_0 > 0$

Comments

- Recall: $-a_0 < a_2$ & $a_0 < a_2$ Sufficient for complex conjugate roots and only necessary for real roots.
- Real roots: substituting the three conditions in the z-domain characteristic polynomial gives roots between –1 and +1.

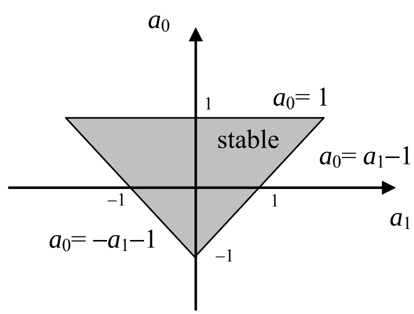
Stable Parameter Range for 2nd Order z-polynomial

Conditions with $a_2 = 1$:

$1 - a_1 + a_0 > 0$

$1 - a_0 > 0$

$1 + a_1 + a_0 > 0$



Jury Test

The roots of the polynomial

$$F(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, a_n > 0$$

are inside the unit circle if and only if

(1) $F(1) > 0$

(2) $(-1)^n F(-1) > 0$

(3) $|a_0| < a_n$

(4) $|b_0| > |b_{n-1}|$

(5) $|c_0| > |c_{n-2}|$

⋮

(n+1) $|r_0| > |r_2|$

Table Entries

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad k = 0,1,\dots,n-1$$

$$c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ b_{n-1} & b_k \end{vmatrix}, \qquad k = 0,1,\dots,n-2$$

$$\vdots$$

$$r_0 = \begin{vmatrix} s_0 & s_3 \\ s_3 & s_0 \end{vmatrix}, \qquad r_1 = \begin{vmatrix} s_0 & s_2 \\ s_3 & s_1 \end{vmatrix}$$

$$r_2 = \begin{vmatrix} s_0 & s_1 \\ s_3 & s_2 \end{vmatrix}$$

Jury Table

Row	z^0	z^1	z^2	…	z^{n-k}	…	z^{n-1}	z^n
1	a_0	a_1	a_2	…	a_{n-k}	…	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	…	a_k	…	a_1	a_0
3	b_0	b_1	b_2	…	b_{n-k}	…	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	…	b_k	…	b_0	
5	c_0	c_1	c_2	…	…	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	…	…	c_0		
.	.	.	.	…	…			
.	.	.	.	…	…			
.	.	.	.	…	…			
$2n-5$	s_0	s_1	s_2	s_3				
$2n-4$	s_3	s_2	s_1	s_0				
$2n-3$	r_0	r_1	r_2					

Remarks

1. First row of Jury table: list the coefficients of the polynomial $F(z)$ in order of increasing power of z.
2. Number of rows of table $2n-3$ is always odd and the coefficients of each even row are the same as the odd row directly above it with the order of the coefficients reversed.
3. There are $n+1$ conditions for $n+1$ coefficients of $F(z)$.
4. Conditions 3 through $n+1$ are calculated using the coefficient of the first column of the Jury table, together with the last coefficient of the last row.
5. The middle coefficient of the last row is never used and need not be calculated.
6. Conditions (1) and (2) are calculated from $F(z)$ directly. If one of the first two conditions is violated, $F(z)$ has roots on or outside the unit circle (no need to construct the Jury table or test the remaining conditions).

Remarks (cont.)

7- Condition (3) with $a_n = 1$, requires the constant term of the polynomial to be less than unity in magnitude. The constant term is simply the product of the roots and must be smaller than unity for all the roots to be inside the unit circle.

8 For first and second order systems, the Jury stability conditions reduce to the conditions derived earlier.

9- For higher order systems, applying the Jury test by hand is laborious and it is preferable to test the stability of a polynomial $F(z)$ using a CAD package.

10- If the coefficients of the polynomial are functions of system parameters, the Jury test can be used to obtain the stable ranges of the system parameters.

Example

Test the stability of the polynomial

$$F(z) = z^5 + 2.6z^4 - 0.56z^3 - 2.05z^2 + 0.0775z + 0.35$$

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	–2.05	–0.56	2.6	1
2	1	2.6	–0.56	–2.05	0.0775	0.35
3	–0.8775	–2.5729	–0.1575	1.854	0.8325	
4	0.8325	1.854	–0.1575	–2.5729	–0.8775	
5	0.0770	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.0770		
7	–0.2593	–0.0837	–0.3472			

Solution

(1) $F(1) = 1 + 2.6 - 0.56 - 2.05 + 0.0775 + 0.35 = 1.4175 > 0$

(2) $(-1)^5 F(-1) = (-1)(-1 + 2.6 + 0.56 - 2.05 - 0.0775 + 0.35)$
$= -0.3825 < 0$

(3) $|0.35| < 1$

(4) $|-0.8775| > |0.8325|$

(5) $|0.0770| < |0.5151|$

(6) $|-0.2593| < |-0.3472|$

- Conditions (2), (5) & (6) violated .
- Condition (2) is sufficient to conclude instability.

No. of conditions violated ≠ No. of roots outside the unit circle.

- Polynomial has roots on or outside the unit circle.
  $F(z) = (z - 0.7)(z - 0.5)(z + 0.5)(z + 0.8)(z + 2.5)$
- Root at -2.5 outside the unit circle.

Example 4.6

Find the stable range of the gain K for the unity feedback digital control system with analog plant

$$G(s) = \frac{K}{s+3}$$

with DAC and ADC if the sampling period is 0.02 s.

Solution

- Transfer function of analog subsystem, ADC and DAC

$$G_{ZAS}(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$
$$= \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{K}{s(s+3)}\right\}$$

- Partial fraction expansion

$$\frac{K}{s(s+3)} = \frac{K}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

- Transfer function ($T = 0.02\ s$)

$$G_{ZAS}(z) = \frac{1.9412 \times 10^{-2} K}{z - 0.9418}$$

Closed-loop System

- Unity feedback, closed-loop characteristic equation $1 + G_{ZAS}(z) = 0$

$$z - 0.9418 + 1.9412 \times 10^{-2} K = 0$$

- Stability conditions $|a_0| < 1 \Rightarrow -1 < a_0 < 1$

$$0.9418 - 1.9412 \times 10^{-2} K < 1$$

$$-0.9418 + 1.9412 \times 10^{-2} K < 1$$

- Hence, the stable range of K is

$$-3 < K < 100.03$$

Example

Find the stable range of the gain K for the digital position control system with the analog plant transfer function

$$G(s) = \frac{K}{s(s+10)}$$

and with DAC and ADC if $T = 0.05\ s$.

Solution

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{K}{s^2(s+10)}\right\}$$

- Partial fraction expansion

$$\frac{K}{s^2(s+10)} = 0.01K\left[\frac{10}{s^2} - \frac{1}{s} + \frac{1}{s+10}\right]$$

- Transfer function ($T = 0.05\ s$)

$$G_{ZAS}(z) = 1.0653 \times 10^{-2} \frac{K(z+0.8467)}{(z-1)(z-0.6065)}$$

C.L. Characteristic Equation

$$G_{ZAS}(z) = 1.0653 \times 10^{-2} \frac{K(z+0.8467)}{(z-1)(z-0.6065)}$$

$$1 + G_{ZAS}(z) = 0$$

$$(z-1)(z-0.6065) + 1.0653 \times 10^{-2} K(z+0.8467)$$

$$= z^2 + (1.0653 \times 10^{-2} K - 1.6065)z + 0.6065$$

$$+9.02 \times 10^{-3} K = 0$$

Stability Testing

$$F(z) = z^2 + \left(1.0653 \times 10^{-2}K \;-1.6065\right)z$$
$$+0.6065 + 9.02 \times 10^{-3}K$$

1) $F(1) = 1 + 1.0653 \times 10^{-2}K \;-1.6065 + 0.6065$
$+\,9.02{\times}10^{-3}K \;>\; 0, \qquad \forall K \;>\; 0$

2) $F(-1) = 1-(1.0653{\times}10^{-2}\,K{-}1.6065) + 0.6065$
$+9.02{\times}10^{-3}K \;>\; 0 \Leftrightarrow K \;<\; 1967.582$

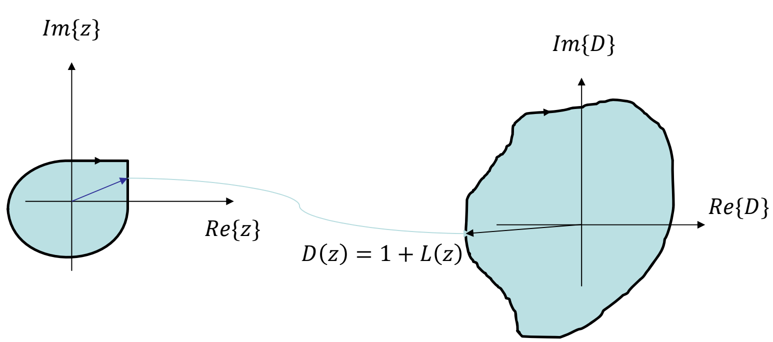
3) $|a_0| < a_n,\;\; |\,0.6065 \;+\; 0.0902\,K\,| < 1$
$\Leftrightarrow \;+\;(0.6065 \;+\; 0.0902\,K\,) < 1$
$\&\; -(0.6065 \;+\; 0.0902\,K\,) \;< 1$
$\Leftrightarrow -178.104 \;<\; K \;<\; 43.6199$

- The stable range is $0 \;<\; K \;<\; 43.6199$

Nyquist Criterion

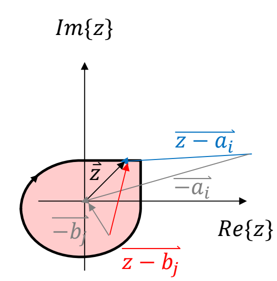
- Given the number of unstable open-loop poles.
- Count the number of unstable closed-loop poles.
- Follow a closed contour and substitute each point z in $D(z)$ (vector)
- Plot the resulting closed curve.
- Count the number of counterclockwise turns N of the vector $D(z)$ (clockwise is $-N$)

Mapping



Count the number of counterclockwise turns N of the vector $D(z)$ (clockwise is $-N$)

Inside/Outside Contour



$$D(z) = 1 + L(z) = \frac{N_L + D_L}{D_L} = \frac{(z-a_1)(z-a_2)\ldots(z-a_n)}{(z-b_1)(z-b_2)\ldots(z-b_n)}$$

a_i =closed-loop poles, b_i =open-loop poles

Net rotation: zero for outside, 1 turn for inside

Thm. 4-7 Nyquist Criterion

- N =no. of counterclockwise encirclements of the point (–1,0) for a loop gain $L(z)$ when traversing the stability contour (i.e. $-N$ clockwise)
- $L(z)$ has P open-loop poles inside the contour.
- Then the system has closed-loop poles outside the unit circle with given by $(-N = Z - P)$, or

$$Z = (-N) + P$$

Corollary: An open-loop stable system is closed-loop stable **iff** the Nyquist plot does not encircle the point (–1,0) (i.e. $-N = 0$)

Proof Outline

- Traverse contour clockwise and substitute in $L(z)$ to produce a plot.

$$1 + L(z) = 1 + \frac{N_L}{D_L} = \frac{D_L + N_L}{D_L}$$

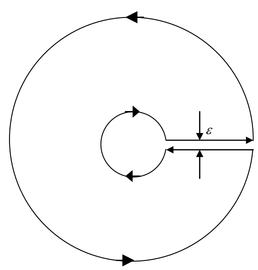
$$= \frac{\text{c.l. characteristic polynomial}}{\text{o.l. characteristic polynomial}}$$

- Net angle change: $+2\pi$/zero, -2π/pole inside the contour and zero for those outside it.
- One encirclement = 2π

$$(-N) = Z - P$$

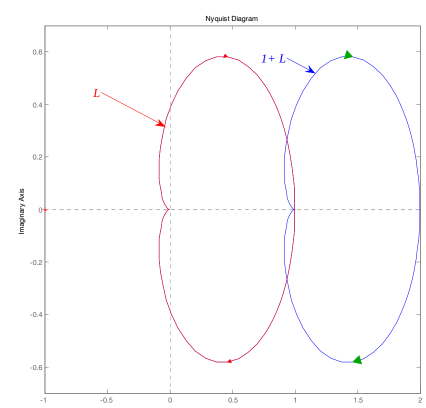
Z ($resp.\ P$)= number of c.l. (resp. o.l.) poles inside the contour.

Contour for Stability Determination



- On the unit circle, $L(e^{j\omega T})$ is the frequency response for angles $\omega T \in [0, \pi]$, or $\omega \in [0, \pi/T] = [0, \omega_s/2]$.
- $L(e^{-j\omega T})$ complex conjugate of $L(e^{j\omega T})$ (mirror image)
- Points on the large circle map to zero or to a single point.
- For ε infinitesimal, angle contributions of values on the straight line portions close to the real axis cancel.

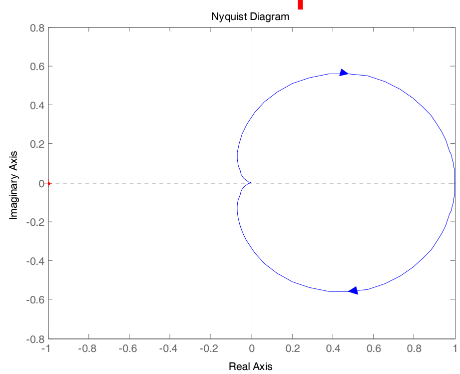
Shift by (-1)



No. encirclements of origin for $1 + L(z)$

= No. of encirclements of $(-1,0)$ of $L(z)$

Example



$$G_{ZAS}(z) = 10^{-5}\frac{4.95z + 4.901}{z^2 - 1.97z + 0.9704}$$

$N = 0$, no open-loop poles outside the unit circle: system is closed-loop stable.

Modified Contour

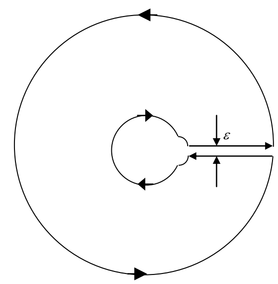
Modified contour

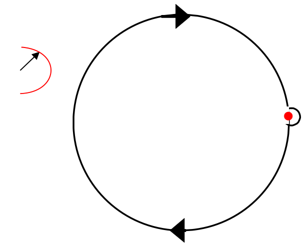
- Avoid pole at unity
- Maps to large semicircle.

$$G(z) \approx \frac{K}{(z-1)^m}$$

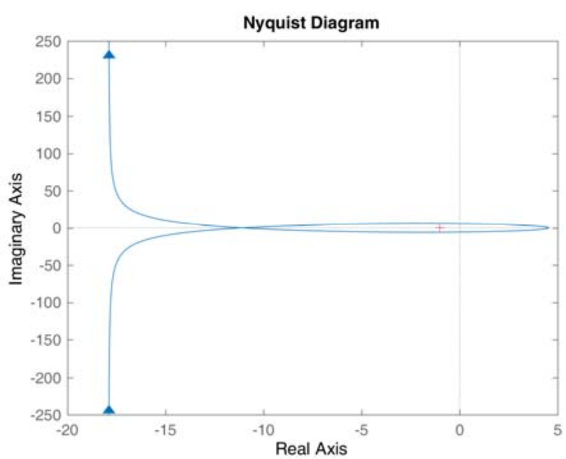
- Small semicircle: $z - 1 = \epsilon e^{j\theta}$
- Pole at 1 mapped to large clockwise semicircle

Denominator angle: net effect $= m$ counterclockwise semicircles





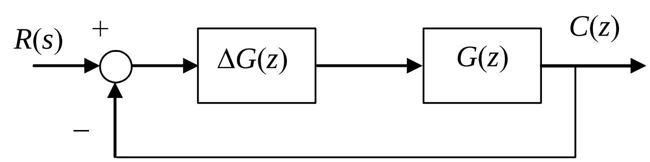
>>gz=10/(z-1)/(z-0.1)



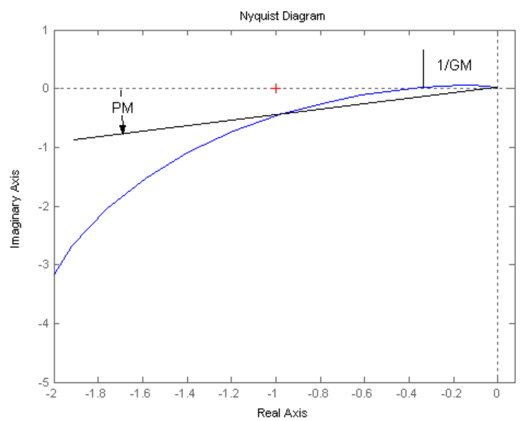
Phase Margin and Gain Margin

Gain Margin: gain perturbation that makes the system marginally stable.

Phase Margin: negative phase perturbation that makes the system marginally stable.



Nyquist Plot: PM & GM



MATLAB Plots and Margins

>> nyquist(gd) % Nyquist plot.

>> bode(gd) % Bode plot.

>> **[gm,pm]=margin(gd) % Find PM & GM**

>> margin(gd) % GM & PM on Bode plot

Nyquist plot: click on plot and select

Characteristics

All stability margins

Example 4.10

Digital control system for the furnace model

$G(s) = \frac{1}{s^2+3s+1}, T = 0.01\ s$

Discrete-time first order actuator

$$G_a(z) = \frac{0.9516}{z - 0.9048}$$

1. Determine the closed-loop stability.
2. How does an amplifier of gain $K = 5$ affect closed-loop stability?

Solution

$$G_a(z)G_{ZAS}(z) = 10^{-5}\frac{4.711z + 4.644}{z^3 - 2.875z^2 + 2.753z - 0.8781}$$

- Using MATLAB

>> gd=c2d(g,0.01)

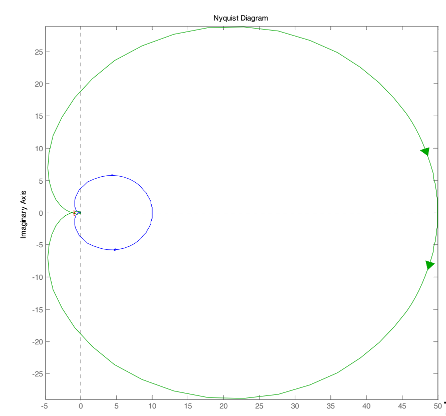
>> gtd=gd*ga;

>> nyquist(gtd)

>> hold on

>> nyquist(5*gtd)

- Change scale



GM & PM

$K = 1, N = 0$ stable

$K = 5$

$Z = (-N) + P$

$= 2 + 0$

>>[gm,pm]=margin(gtd)

gm =

3.4817

pm =

37.5426