Z-Domain Root Locus

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Outline

• Root locus plots in z-plane.
• Pole locations and time response.
• Z-plane contours.
• Proportional control.

Closed-loop Characteristic Equation

1 + C(z)G_{ZAS}(z) = 0
1 + KL(z) = 0

C(z)= controller transfer function
G_{ZAS}(z)= transfer function of DAC, analog subsystem, and sampler
L(z)= loop gain
K= gain

Observations

• Identical equation to s-domain equation with $s$ replaced by $z$.
• All the rules derived for s-domain are applicable and can be used to obtain z-domain root locus plots.
• The plots can also be obtained using the root locus plots of most CAD programs (MATLAB = rlocus)
Example 6.1

- First order type 1 system with loop gain
  \[ L(z) = \frac{1}{z - 1} \]
- Obtain the root locus plot.
- Obtain the critical gain.

Solution

- Root locus rules give plot (MATLAB rlocus).
- Root locus: real axis locus between pole and zero.
- For a stable discrete system, real axis loci must lie between (1,0) and (−1,0) in the z-plane.
- Critical gain \( K_{cr} \) is at point (−1,0).

Example 6.2

- Second order type 1 system with loop gain
  \[ L(z) = \frac{1}{(z - 1)(z - 0.5)} \]
- Obtain the root locus plot
- Obtain the critical gain.
Solution

- Use root locus rules.
- RL: like Example 5.1(i) but in RHP.
- Breakaway point: \( z_b = (1 + 0.5)/2 = 0.75 \)
- \( K_{cr} \) (critical) intersection of RL & unit circle.
- Closed loop characteristic equation
  \( (z - 1)(z - 0.5) + K = z^2 - 1.5z + K + 0.5 = 0 \)

On the unit circle, \( |z| = 1 \) (complex conjugate)

\[
|z_{1,2}| = K_{cr} + 0.5 = 1
\Rightarrow K_{cr} = 0.5, \quad z_{1,2} = 0.75 \pm j0.661
\]

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### z-Domain Pole Locations & Associated Temporal Sequences

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### Time Functions & Real Poles

<table>
<thead>
<tr>
<th>Continuous</th>
<th>Laplace Transform</th>
<th>Sampled</th>
<th>z-Transform</th>
</tr>
</thead>
</table>
| \( f(t) \) | \( F(s) = \frac{1}{s + \alpha} \) | \( f(kT) = \begin{cases} 
  e^{-\alpha T}, & t \geq 0 \\
  0, & t < 0 
\end{cases} \) | \( F(z) = \frac{z}{z - e^{-\alpha T}} \) |
Time Functions & Complex Conjugate Poles

<table>
<thead>
<tr>
<th>Continuous</th>
<th>( f(t) = \begin{cases} e^{-\xi \omega_n t} \sin(\omega_d t), &amp; t \geq 0 \ 0, &amp; t &lt; 0 \end{cases} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace Transform</td>
<td>( F(s) = \frac{\omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} )</td>
</tr>
<tr>
<td>Sampled</td>
<td>( f(kT) = \begin{cases} e^{-\xi \omega_n kT} \sin(\omega_d kT), &amp; k \geq 0 \ 0, &amp; k &lt; 0 \end{cases} )</td>
</tr>
<tr>
<td>( z )-Transform</td>
<td>( F(z) = \frac{\sin(\omega_d T)e^{-\xi \omega_n T}z}{z^2 - 2 \cos(\omega_d T)e^{-\xi \omega_n T} + e^{-2\xi \omega_n T}} )</td>
</tr>
</tbody>
</table>

2nd Order Underdamped System

- **z-domain characteristic polynomial**
  \( (z - e^{(-\xi \omega_n + j \omega_d)T})(z - e^{(-\xi \omega_n - j \omega_d)T}) = z^2 - 2 \cos(\omega_d T)e^{-\xi \omega_n T}z + e^{-2\xi \omega_n T} \)

- **Poles**
  \( z_{1,2} = e^{(-\xi \omega_n \pm j \omega_d)T} = e^{-\xi \omega_n T} \angle \pm \omega_d T \)

Observation

If the Laplace transform \( F(s) \) of a continuous-time function \( f(t) \) has a pole \( p_s \), then the \( z \)-transform \( F(z) \) of its sampled counterpart \( f(kT) \), with sampling period \( T \) has a pole at \( p_z = e^{p_s T} \)

Primary Strip

\[
\begin{align*}
\omega & \\
\text{Primary Strip} & \\
\omega/2 & \\
-\omega/2 & \\
\sigma & \\
\end{align*}
\]

\[
p_z = e^{p_s T} = e^{\sigma T}e^{j\omega_d T} = e^{\sigma T}e^{j(\omega_d T + 2k\pi)}, k = 0,1,2, ...
\]
Pole Contours in the s-Domain and the z-Domain

Poles $s_{1,2} = \sigma \pm j \omega_d$, $z_{1,2} = e^{\sigma T} \angle \pm \omega_d T$

<table>
<thead>
<tr>
<th>Contour</th>
<th>s-Domain Poles</th>
<th>Contour</th>
<th>z-Domain Poles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$ constant</td>
<td>vertical line</td>
<td>$</td>
<td>z</td>
</tr>
<tr>
<td>$\omega_d$ constant</td>
<td>horizontal line</td>
<td>$\angle z$ constant</td>
<td>radial line</td>
</tr>
</tbody>
</table>

Constant $\sigma$ Contours

Constant $\omega_d$ Contours

Constant $\zeta$ Contours

Poles $z_{1,2} = e^{-\zeta \omega_n T} \angle \pm \omega_n T \sqrt{1 - \zeta^2}$

- Logarithmic spirals that get smaller for larger values of $\zeta$.
- The spirals are defined by the equation
  
  \[ |z| = e^{-\zeta \theta / \sqrt{1 - \zeta^2}} = e^{-\zeta \left( \frac{\pi \theta}{180^\circ} \right) / \sqrt{1 - \zeta^2}} \]

  $|z|$ = magnitude of pole
  $\theta$ = angle of pole
Constant $\omega_n$ Contours

$$|z| = e^{-\sqrt{(\omega_nT)^2-\theta^2}}$$

$|z|$= magnitude of pole

$\theta$= angle of pole

To obtain the expression, eliminate $\zeta$ from

$$z_{1,2} = e^{-\zeta \omega_nT} \pm \omega_n T \sqrt{1 - \zeta^2}$$

Characteristics of Log Spirals

1. Two spiral for each $\zeta$ value, corresponding to $\pm \theta$:
   
   - $\theta$ spiral below real axis=mirror image of $+\theta$ spiral

2. For any spiral, $|z|$ drops logarithmically with $\theta$ increase.

3. At the same angle, increasing $\zeta$ gives smaller $|z|$, i.e. spirals are smaller for larger $\zeta$ values.

4. All spirals start at $\theta = 0, |z| = 1$ but end at different points.

5. When given $\zeta$ and $|z|$, obtain $\theta$ by substituting in the equation

$$\theta = \frac{\sqrt{1 - \zeta^2}}{\zeta} \cdot |\ln(|z|)|$$

MATLAB

```matlab
>> g=tf(num, den, T) % sampling period T
>> rlocus(g) % Root locus plot
>> zgrid(zeta, wn) % Plot contours
% zeta = vector of damping ratios
% wn = vector of undamped natural frequencies
```
z-Domain Design Specifications

- Similar to those for s-domain design.
- Often approximate values based on continuous time definitions.
- Allow selection of pole locations for z-domain design.

Time Constant

- Time constant of exponential decay for the continuous envelope of sampled waveform.
  \[ \tau = \frac{1}{\zeta \omega_n} \]
- Not necessarily equal to a specified percentage of the final value after one time constant

Settling Time

- Period after which the envelope of the sampled waveform stays within a specified percentage (usually 2%) of the final value.
- Multiple of time constant depending on the specified percentage.
- Settling time for a 2% specification
  \[ T_s = \frac{4}{\zeta \omega_n} \]

Other Specification

- Frequency of Oscillations \( \omega_d \): angle of the dominant complex conjugate poles divided by the sampling period.
- Other design criteria such as the percentage overshoot, \( \zeta, \omega_n \), defined analogously to the continuous case.
- As in analog design, select a dominant closed-loop pair in the complex plane to obtain a satisfactory time response.
- Analytical design: possible for low order systems but more difficult than analog design.
Example 6.3

Design a proportional controller for the digital system with sampling period $T=0.1\text{s}$ to obtain

a) $\omega_d = 5\text{ rad/s}$

b) A time constant of 0.5 s

c) $\zeta = 0.7$

$$L(z) = \frac{1}{(z - 1)(z - 0.5)}$$

Solution

- Obtain the results with calculator or MATLAB.
- Use MATLAB: `rlocus`
  
  (a) $\omega_d = 5\text{ rad/s}$:
  
  angle of the pole $= \omega_d T$
  
  $= 5 \times 0.1 = 0.5\text{ rad} = 28.65^\circ$

  (b) $\tau = 0.5\text{ s}$:
  
  $1$/(time constant)$= \zeta \omega_n = 1/0.5 = 2\text{ rad/s}$
  
  pole magnitude $= \exp(-\zeta \omega_n T) = 0.82$

  (c) $\zeta = 0.7$:
  
  Use $\zeta$ directly to get the results of Table 6.1.

- Sampled step response: MATLAB command `step`
- Higher gain designs have a low $\zeta$ (oscillatory response).

## P-Control Design Results

<table>
<thead>
<tr>
<th>Design</th>
<th>Gain</th>
<th>$\zeta$</th>
<th>$\omega_n\text{ rad/s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_d = 5\text{ rad/s}$</td>
<td>0.23</td>
<td>0.3</td>
<td>5.24</td>
</tr>
<tr>
<td>$\tau = 0.5\text{ s}$</td>
<td>0.17</td>
<td>0.4</td>
<td>4.60</td>
</tr>
<tr>
<td>$\zeta = 0.7$</td>
<td>0.10</td>
<td>0.7</td>
<td>3.63</td>
</tr>
</tbody>
</table>

## Time Response Plots

Designs of Table 6.1 (a) = •, (b) = ●, (c) = +.
Analytical Design

\[ L(z) = \frac{1}{(z - 1)(z - 0.5)} \]

- Closed-loop characteristic equation

\[ z^2 - 1.5z + 0.5 + K = z^2 - 2 \cos(\omega_d T) e^{-\zeta \omega_n T} z + e^{-2\zeta \omega_n T} \]

- Equating coefficients

\[
\begin{align*}
\zeta \omega_n &= 1/	au = 1/0.5 = 2 \text{ rad/s} \\
z^1: \quad 1.5 &= 2 \cos(\omega_d T) e^{-\zeta \omega_n T} \\
\omega_d &= \frac{1}{T} \cos^{-1} \left( \frac{1.5 e^{\zeta \omega_n T}}{2} \right) \\
&= 10 \cos^{-1} \left( 0.75 e^{0.2} \right) \approx 4.127 \text{ rad/s} \\
K &= e^{-2\zeta \omega_n T} - 0.5 = e^{-2 \times 1.571 \times 0.1} - 0.5 \approx 0.23 \\
\end{align*}
\]

Design (a) \( \omega_d = 5, T = 0.1 \)

- \( z^1 \) equation: \[ 1.5 = 2 \cos(\omega_d T) e^{-\zeta \omega_n T} \]

\[
\begin{align*}
\zeta \omega_n &= -\frac{1}{T} \ln \left( \frac{1.5}{2 \cos(\omega_d T)} \right) \\
&= -10 \ln \left( \frac{1.5}{2 \cos(0.5)} \right) \\
&\approx 1.571 \\
\omega_d &= \omega_n (1 - \zeta^2) = 25 \\
\frac{\omega_d^2}{(\zeta \omega_n)^2} &= \frac{1 - \zeta^2}{\zeta^2} = \frac{25}{(1.571)^2} \Rightarrow \zeta = 0.3, \omega_n = 5.24 \frac{\text{rad}}{\text{s}} \\
\end{align*}
\]

- \( z^0 \) equation: \[ K + 0.5 = e^{-2\zeta \omega_n T} \]

\[
\begin{align*}
K &= e^{-2\zeta \omega_n T} - 0.5 = e^{-2 \times 1.571 \times 0.1} - 0.5 \approx 0.23 \\
\end{align*}
\]

Design (b) \( \tau = 0.5 \text{ s} \)

\[
\begin{align*}
\zeta \omega_n &= 1/\tau = 1/0.5 = 2 \text{ rad/s} \\
z^1 \text{ equation: } 1.5 &= 2 \cos(\omega_d T) e^{-\zeta \omega_n T} \\
\omega_d &= \frac{1}{T} \cos^{-1} \left( \frac{1.5 e^{\zeta \omega_n T}}{2} \right) \\
&= 10 \cos^{-1} \left( 0.75 e^{0.2} \right) \approx 4.127 \text{ rad/s} \\
K &= e^{-2\zeta \omega_n T} - 0.5 = e^{-2 \times 1.571 \times 0.1} - 0.5 \approx 0.23 \\
\end{align*}
\]

Design (c) \( \zeta = 0.7 \)

- \( z^1 \) equation: \[ 1.5 = 2 \cos(\omega_d T) e^{-\zeta \omega_n T} \]

\[
\begin{align*}
1.5 &= 2 \cos(0.0714 \omega_n) e^{-0.07 \omega_n} \\
\omega_n &= 3.63 \text{ rad/s} \\
K &= e^{-2\zeta \omega_n T} - 0.5 = e^{-0.14 \times 3.63} - 0.5 \approx 0.1 \\
\end{align*}
\]

- \( z^0 \) equation: \[ K + 0.5 = e^{-2\zeta \omega_n T} \]

\[
\begin{align*}
K &= e^{-2\zeta \omega_n T} - 0.5 = e^{-2 \times 1.571 \times 0.1} - 0.5 \approx 0.23 \\
\end{align*}
\]

Graphically: draw root locus and a segment of the constant \( \zeta \) spiral and find their intersection.

(Rough results and the solution is difficult without MATLAB for all but a few simple root loci).
Example 6.4

Design a proportional controller for the unity feedback digital control system with a sampling period $T = 0.1$ s to obtain

a) $e(\infty)$ due to ramp = 10%

b) $\zeta = 0.7$

$$G(s) = \frac{1}{s(s + 5)}$$

Solution

$$G_{ZAS}(z) = 4.261 \times 10^{-3} \frac{z + 0.847}{(z - 1)(z - 0.606)}$$

- Closed-loop characteristic equation

$$1 + KG_{ZAS}(z)$$

$$= z^2 - (1.606 - 4.261 \times 10^{-3}K)z + 0.606 - 3.608 \times 10^{-3}K$$

- Equation involves three parameters $\zeta$, $\omega_n$ & $K$

- Equating coefficients yields two equations

- Evaluate two unknowns and obtain the third from the design specification.

(a) Design for $e(\infty)\% = 10$

- Velocity error constant $K_v$ (Type 1)

$$K_v = \frac{1}{T} \left( \frac{z - 1}{z} \right) KG_{ZAS}(z) \bigg|_{z=1}$$

$$= 10K \frac{4.261 \times 10^{-3}(1 + 0.847)}{1 - 0.606} = \frac{K}{5}$$

- Same $K_v$ as for the analog proportional control system: 10% steady-state error due to ramp.

$$K = 50$$

Root locus for $K = 50$
b) Design for $\zeta = 0.7$

- Difficult analytical solution for constant $\zeta$
- MATLAB: move cursor to $\zeta = 0.7 \Rightarrow K = 10$
- $K_{cr} = 109$, stable system at $K = 50$ and $K = 10$.
- Design specs. are met for both (a) and (b).
- Must check other design criteria.
- For (a), $K = 50$, $\zeta = 0.18$ (highly oscillatory)
- For (b), $K = 10$, $e(\infty)\% = 50\%$ due to unit ramp.
- P-control cannot provide good steady-state error together with good transient response.