1. Find the general solution (if solutions exist) of each of the following linear Diophantine equations:

1. \(2x + 3y = 4\)

\(\gcd(2, 3) = 1\) which divides 4. So the linear Diophantine equation have integer solutions. In order to find them, we use Euclid’s division algorithm and get \(3 = 2 \times 1 + 1\). This means \(1 = 2 \times (-1) + 3 \times 1\); therefore

\[2 \times (-4) + 3 \times 4 = 4.\]

We have obtained a solution of this equation \(x = -4, y = 4\). Now by Theorem 2.9, we can write down the general solutions as

\[
\begin{align*}
x &= -4 + 3t \\
y &= 4 - 2t
\end{align*}
\]

where \(t\) is an arbitrary integer.

2. \(15x + 51y = 41\)

Since \(\gcd(15, 51) = 3\) which does not divide 41. So the linear diophantine equation does not have any integer solutions.

2. Tom pays $1.43 for some apples and pears. If pears cost $0.17 each, and apples, $0.15 each, how many of each did Tom buy?

Below the lines I will present three solutions. The congruence method in Solution 3 had not yet been talked about in class until Feb 11. But I decided to include it here for comparison.

*Solution 1:*
Suppose Tom buys $x$ apples and $y$ pears. Then $0.15x + 0.17y = 1.43$, which can be written as

\[15x + 17y = 143\]

with integer coefficients. Now we can solve this linear diophantine equation with the standard method. Namely

\[17 = 15 \cdot 1 + 2\]
\[15 = 2 \cdot 7 + 1\]

So $1 = 15 - 2 \cdot 7 = 15 - (17 - 15) \cdot 7 = 15 \cdot 8 + 17 \cdot (-7)$. We then have

\[15 \cdot 1144 + 17 \cdot (-1001) = 143.\]

The solution $x = 1144, y = -1001$ is not sensible for our problem, since $x, y$ cannot be negative. Instead, we find the general solution

\[
\begin{aligned}
x &= 1144 - 17t \\
y &= -1001 + 15t
\end{aligned}
\]

In order for $y \geq 0$, we must have $t \geq 67$. But in order for $x \geq 0$, we must have $t \leq 67$. So $t = 67$. This gives

\[
\begin{aligned}
x &= 1144 - 17 \cdot 67 = 5 \\
y &= -1001 + 15 \cdot 67 = 4
\end{aligned}
\]

So Tom bought 5 apples and 4 pears.

**Solution 2:**

Solution 1 uses our general theory about linear diophantine equations, which represents the standard solution. However, pertaining to the special feature of the problem under discussion, one may find shortcuts. Since either $x$ or $y$ cannot be negative. We must have $17y \leq 143$, which means $y \leq 143/17$. This only leaves a fews choices for $y$, namely $y$ can only be one of the following nine numbers, 0, 1, 2, 3, 4, 5, 6, 7, 8. One can then eliminate every number but 4 by plugging them into the
equation and solving for $x$ (which has to be an integer). This gives another possible solution; but I don’t really recommend it since as the coefficients become large this method needs a lot more computations than Solution 1 and is very inefficient.

**Solution 3:**
We may as well solve this problem using our congruence method in Chapter 4. The diophantine equation $15x + 17y = 143$ can be restated as

$$17y \equiv 143 \pmod{15}. \quad (1)$$

(or we may as well write $15x \equiv 143 \pmod{17}$. But think about why we prefer the former rather than the latter.) Now notice that $17 \equiv 2 \pmod{15}$ and $143 \equiv 8 \pmod{15}$. Equation (1) becomes

$$2y \equiv 8 \pmod{15}.$$ 

Observe that $2 \ast 8 \equiv 1 \pmod{15}$ (as we talked about in class, this means that 8 plays the role of $\frac{1}{2}$ in the modulo 15 world). So we have

$$8 \ast 2y \equiv 8 \ast 8 \pmod{15}$$

which can be reduced to

$$y \equiv 4 \pmod{15}.$$ 

So $y$ can only possibly attain values like 4, 19, 34, 49, $\cdots$. But clearly in our problem $y$ has to be 4, since all the other values of $y$ would have made the corresponding $x$ negative which makes no sense for the problem under discussion.