Proof by contradiction

Want to prove a proposition $P$

Step 1. Assume $P$ is false (i.e., $\neg P$ is true)

Step 2. Deduce a contradiction from the assumption in Step 1

Step 3. Claim that $P$ is true.

Ex 1. Show that there are infinitely many primes.

Proof. Suppose that there are only finitely many

Ex 2. Show that $\sqrt{2}$ is irrational.
Existence proof vs Constructive proof.

Pigeonhole principle: If $n$ items are put into $m$ containers, with $n > m$, then at least one container must contain more than one item.

Applications:

Ex.1. In a party of at least 367 people, there must be two one pair who shares the same birthday.

Q1.1 How many people are needed to guarantee a triple who shares the same birthday?

Ex.1.2 In a party of six people, there must be three of them who mutually know each other or who mutually don't know each other.
Q.2: How many people are needed to guarantee there exist four of them who mutually know each other or who mutually don't know each other?

\[ \text{A \rightarrow B} \quad \text{A, B know each other.} \]
\[ \text{A \rightarrow \rightarrow B} \quad \text{A, B don't know each other.} \]

By pigeonhole principle

Let's focus on the first case, since the second one is analogous. If \( B \overset{\text{know}}{\rightarrow} C \text{ or } C \overset{\text{know}}{\rightarrow} D \), or \( B \overset{\text{known}}{\rightarrow} D \), then we are done. If none of those three happen, then \( B \overset{\text{known}}{\rightarrow} C \text{ and } C \overset{\text{known}}{\rightarrow} D \text{ and } B \overset{\text{known}}{\rightarrow} D \), we are done anyway!
Dirichlet's Theorem 1.1.

For every real number $\alpha$ and any positive integer $Q$, there exists two integers $p$ and $q$ such that $1 \leq q \leq Q$

and

$$0 < q - \frac{p}{q} < \frac{1}{2Q}$$

Proof.

By mathematical induction, let $S_n = 1+2+3+\ldots+n$. Then $S_n = \frac{n(n+1)}{2}$.

Base case

17 = $1 \times 10 + 7 \times 1 = 1 \times 2^2 + 1 \times 1 = 10001$ (base 2)

217 = $2 \times 100 + 1 \times 10 + 7 \times 1 = 1 \times 10^3 + 0 \times 10^2 + 3 \times 25 + 0 \times 3 \times 5 + 2$ (base 5)

Thus, any integer greater than 1 can serve as a base for representing positive integers.