when \( i = 1 \), \( k = r - 1 \) and when \( i = r \), \( k = 0 \). Thus, substituting \( r - k \) for \( i \) on the left side of (1.13), we obtain
\[
\sum_{i=1}^{r} i^2 = \sum_{k=0}^{r-1} (r - k)^2 = \sum_{k=0}^{r-1} (r - k)^2
\]
(1.15)
since a sum is the same if we sum either forward or backward. In this case the substitution seems to make the sum more involved, but the idea is important and the device of substitution of indices is often useful in manipulating with sums and products.

**EXERCISES 1.1**

1. Write out the following sums.
   (a) \( \sum_{i=1}^{5} (2i - 1) \)  
   (b) \( \sum_{i=0}^{6} \sin ix \)  
   (c) \( \sum_{i=0}^{n} f(i) \)  
   (d) \( \sum_{j=1}^{n} \frac{2}{j(j+1)} \)  
   (e) \( \sum_{k=5}^{10} 3 \)  
   (f) \( \sum_{i=3}^{3} \frac{3}{i} \)

2. Use the change of indices \( i = j + 1 \) to rewrite the summations in Exercise 1(a)-(d).

3. Write the following in summation notation.
   (a) \( 2 + 4 + 6 + 8 + 10 \)  
   (b) \( 1 + 8 + 27 + 64 + 125 \)  
   (c) \( 28 + 31 + 34 + 37 + 40 + 43 \)  
   (d) \( n + (n + 2) + (n + 4) + \cdots + (n + 2m) \)

4. Evaluate \( \sum_{i=1}^{n} (a_i - a_{i-1}) \) given that \( a_0 = 0 \).

5. Use the result of Exercise 4 to prove that \( \sum_{i=1}^{n} i = n(n + 1)/2 \).  
   *Hint:* Let \( a_i = i(i + 1)/2 \).

6. Use the result of Exercise 4 to prove that
\[
\sum_{i=1}^{n} i(i + 1) = n(n + 1)(n + 2)/3.
\]

7. With only slight modifications, the equations in Exercises 5 and 6 could have been written in the form
\[
\sum_{i=1}^{n} \binom{i}{1} = \binom{n+1}{2} \quad \text{and} \quad \sum_{i=1}^{n} \binom{i+1}{2} = \binom{n+2}{3},
\]
where
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
is the usual binomial coefficient notation. What more general result do these suggest?

8. Use simple algebraic manipulation to show that

\[
{n \choose k} + {n \choose k+1} = {n+1 \choose k+1}
\]

for all integers \( n \) and \( k \) with \( 0 \leq k \leq n \).

9. Prove that

\[
\sum_{i=1}^{n} \binom{i+k-1}{k} = \binom{n+k}{k+1},
\]

where \( n \) and \( k \) are integers with \( n \geq 1 \) and \( k \geq 0 \). Note that it is customary to set \( \binom{a}{b} = 0 \) for integers \( a \) and \( b \) if \( 0 \leq a < b \).

10. Evaluate \( \sum_{i=0}^{n} \binom{i}{k} \) where \( n \) and \( k \) are nonnegative integers.

**Hint:** Note that \( \binom{i}{k} = 0 \) for \( 0 \leq i < k \) and use the substitution \( i = j + k - 1 \), where \( j \) is the new index of summation.

11. Use the results of Exercises 5 and 6 to derive a formula for

\[
\sum_{i=1}^{n} i^2.
\]

12. Write out the following products:

(a) \( \prod_{j=1}^{4} (2j - 1) \)  
(b) \( \prod_{j=0}^{3} \frac{j}{j+1} \)

(c) \( \prod_{i=p}^{p+n} i \)  
(d) \( \prod_{i=2}^{p} e^i \)

13. Use the change of indices \( i = j - 1 \) to rewrite all the products in Exercise 12.

14. Write the following in product notation.

(a) \( 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \)

(b) \( (-1)^n \cdot n! \)

(c) \( \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{n^2}\right) \)

15. Evaluate \( \Pi_{i=1}^{n} a^i \) and \( \Pi_{i=1}^{n} a^{i(i+1)} \).

16. Evaluate \( \prod_{i=1}^{n} \frac{a_i}{a_{i-1}} \) given that \( a_0 = 1 \).

17. Use the result of Exercise 16 to prove the following:

(a) \( \prod_{i=1}^{n} \frac{i}{i+1} = \frac{1}{n+1} \)  
(b) \( \prod_{i=1}^{n} \left[1 - \frac{1}{(i+1)^2}\right] = \frac{n+2}{2(n+1)} \)
With the data at hand, suffice it to say that one considers examples, tries to identify regular and consistent patterns, and finally formulates general statements which one then endeavors to prove. For example, if we consider the table for a moment, we may note that

\[
\begin{align*}
1 + 2 &= 3, \\
1 + 3 &= 4, \\
2 + 5 &= 7, \\
3 + 8 &= 11, \\
5 + 13 &= 18,
\end{align*}
\]

and be led to guess that

\[F_n + F_{n+2} = L_{n+1}\]  \hspace{1cm} (1.17)

for all \(n \geq 1\). This is not hard to prove, but in this section we devote our attention to guessing, and the proofs will be postponed until Section 1.3 and even later for the more difficult results. As another example, we note that

\[
\begin{align*}
1 + 1 &= 2, \\
1 + 4 &= 5, \\
4 + 9 &= 13, \\
9 + 25 &= 34, \\
25 + 64 &= 89,
\end{align*}
\]

and a moment's reflection suggests that

\[F_n^2 + F_{n+1}^2 = F_{2n+1}, \hspace{1cm} n \geq 1.\]  \hspace{1cm} (1.18)

In the exercises that follow, the reader will find many examples of this sort. Sometimes the guess is easy to make; sometimes it is relatively difficult. In any case, all are interesting and somewhat surprising, and the more difficult ones will provide the greater opportunity for readers to strengthen their mathematical muscles and better prepare themselves for the rigors yet to come in this and other courses.

**EXERCISES 1.2**

1. Guess a formula suggested by each of the following sets of equations.
   (a) \(1 + 4 = 5\) \hspace{1cm} (b) \(1 + 1 = 2\)
   \(3 + 7 = 10\) \hspace{1cm} \(1 + 3 = 4\)
   \(4 + 11 = 15\) \hspace{1cm} \(2 + 4 = 6\)
   \(7 + 18 = 25\) \hspace{1cm} \(3 + 7 = 10\)
   \(11 + 29 = 40\) \hspace{1cm} \(5 + 11 = 16\)
Sec. 1.2 Inductive Reasoning and the Fibonacci Sequence

<table>
<thead>
<tr>
<th>(c)</th>
<th>1 - 1 = 0</th>
<th>(d)</th>
<th>1 \cdot 1 = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 - 1 = 3</td>
<td></td>
<td>1 \cdot 3 = 3</td>
</tr>
<tr>
<td></td>
<td>9 - 4 = 5</td>
<td></td>
<td>2 \cdot 4 = 8</td>
</tr>
<tr>
<td></td>
<td>25 - 9 = 16</td>
<td></td>
<td>3 \cdot 7 = 21</td>
</tr>
<tr>
<td></td>
<td>64 - 25 = 39</td>
<td></td>
<td>5 \cdot 11 = 55</td>
</tr>
</tbody>
</table>

2. Exercise 1(c) suggests that it might be useful to define \( F_0 = 0 \).
   (a) Is this consistent with the pattern established by the defining equations (1.16)?
   (b) Define \( F_{-1}, F_{-2}, F_{-3}, F_{-4}, \) and \( F_{-5} \) in a way that is also consistent with (1.16).
   (c) Can you guess a relation between \( F_n \) and \( F_{-n} \)?

3. Exercise 1(c) also suggests that it is sometimes interesting to look at the differences of squares. What formulas are suggested by the following sets of equations? Note that more than one correct answer may be possible.
   (a) \[
   \begin{align*}
   4 - 1 &= 3 \\
   9 - 1 &= 8 \\
   25 - 4 &= 21 \\
   64 - 9 &= 55 \\
   \end{align*}
   \]
   (b) \[
   \begin{align*}
   9 - 1 &= 8 \\
   25 - 1 &= 24 \\
   64 - 4 &= 60 \\
   169 - 9 &= 160 \\
   \end{align*}
   \]
   (c) \[
   \begin{align*}
   25 - 1 &= 24 \\
   64 - 1 &= 63 \\
   169 - 4 &= 165 \\
   441 - 9 &= 432 \\
   \end{align*}
   \]
   (d) \[
   \begin{align*}
   9 - 1 &= 8 \\
   16 - 9 &= 7 \\
   49 - 16 &= 33 \\
   121 - 49 &= 72 \\
   \end{align*}
   \]
   (e) \[
   \begin{align*}
   16 - 1 &= 15 \\
   49 - 9 &= 40 \\
   121 - 16 &= 105 \\
   324 - 49 &= 275 \\
   \end{align*}
   \]

4. Exercise 3(d) suggests that it might be useful to define \( L_0 = 2 \).
   (a) Is this consistent with equations (1.16)?
   (b) How would you define \( L_{-1}, L_{-2}, \) and \( L_{-3} \)?
   (c) Can you guess a relationship between \( L_{-n} \) and \( L_n \)?

5. What formulas are suggested by the following arrays? Use summation notation in expressing your answer.
   (a) \[
   \begin{align*}
   1 &= 1 \\
   1 + 1 &= 2 \\
   1 + 1 + 2 &= 4 \\
   1 + 1 + 2 + 3 &= 7 \\
   \end{align*}
   \]
   (b) \[
   \begin{align*}
   1 &= 1 \\
   1 + 2 &= 3 \\
   1 + 2 + 5 &= 8 \\
   1 + 2 + 5 + 13 &= 21 \\
   \end{align*}
   \]
   (c) \[
   \begin{align*}
   1 &= 1 \\
   1 + 3 &= 4 \\
   1 + 3 + 8 &= 12 \\
   1 + 3 + 8 + 21 &= 33 \\
   \end{align*}
   \]
   (d) \[
   \begin{align*}
   1 &= 1 \\
   1 - 2 &= -1 \\
   1 - 2 + 5 &= 4 \\
   1 - 2 + 5 - 13 &= -9 \\
   \end{align*}
   \]
**I.4. Third form of the principle of mathematical induction.** Any set of positive integers that contains 1 and 2, and that contains \( k + 2 \) whenever it contains the positive integers \( k \) and \( k + 1 \), contains all positive integers.

In making a proof based on I.4 one would begin by proving the desired result true for \( n = 1 \) and \( n = 2 \). One would then assume that the result is true for \( n = k \) and \( n = k + 1 \), where \( k \) is any fixed but unspecified positive integer and, on the basis of this assumption, prove that the result must also hold for \( n = k + 2 \). Of course, as usual, both parts of the proof are necessary and the second part of the argument must not depend on \( k \) having some particular value.

Finally, how does one decide whether to use I.1, I.2, I.4, or some other variation of mathematical induction? Actually, perhaps on scratch paper, one has to do the second part of the proof to see what is required to get "the next case." Let \( P(n) \) be a proposition about the integer \( n \). If the truth of \( P(k + 1) \) follows from the truth of \( P(k) \), I.1 will do nicely. If the truth of \( P(k + 1) \) depends on the truth of \( P(i) \) for \( 1 \leq i \leq k \), one must use I.2. If the truth of \( P(k + 2) \) follows from the truth of \( P(k) \) and \( P(k + 1) \), then clearly I.4 is needed. But suppose that the truth of \( P(k + 2) \) depends on the truth of \( P(k) \); what then? A moment’s reflection makes it clear that it will suffice to begin by proving that both \( P(1) \) and \( P(2) \) are true. This would be yet another variation of mathematical induction.

**EXERCISES 1.4**

1. Show that none of the following sets contains a least element:
   (a) The set of positive real numbers.
   (b) The set of all integers.
   (c) The set of all real numbers greater than 2.

2. Find the least element in the set
   \[ F = \{1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\}. \]

3. The following equalities are false for most positive integers \( n \). Try to prove each by the method of mathematical induction and show why the method fails. Also, in each case, give a positive integral value for \( n \) for which the equality is false.
   \[
   \begin{align*}
   (a) \quad \sum_{i=1}^{n} (2i + 1) &= n^2 + 2 \\
   (b) \quad \sum_{i=1}^{n} (i + 3) &= n^2 + n + 2 \\
   (c) \quad \sum_{i=1}^{n} 2^{i-1} &= \frac{n(n + 1)}{2} \\
   (d) \quad \sum_{i=1}^{n} (3i - 2) &= n^2 + n + 1
   \end{align*}
   \]
4. Prove that
\[ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \]
for every positive integer \( n \).

5. Prove that
\[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \]
for every positive integer \( n \).

6. Use \( I_1 \) to prove that \( 2^{2n} - 1 \) is divisible by 3 for every positive integer \( n \).
   \textbf{Hint:} For the second part of the proof make your assumption by assuming that there exists an integer \( q \) such that \( 2^{2k} - 1 = 3q \). Then consider
   \[ 2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 3 \cdot 2^{2k} + 2^{2k} - 1 = 3 \cdot 2^{2k} + 3q. \]

7. Prove that \( 2^{2n+1} + 1 \) is divisible by 3 for every positive integer \( n \).

8. Prove that \( f(n) = 3n^3 + 5n^3 + 7n \) is divisible by 15 for every integer \( n \).
   \textbf{Hint:} Note that \( f(-n) = -f(n) \).

9. Prove that \( 3^{2n+1} + 2^{n+2} \) is divisible by 7 for every non-negative integer \( n \).

10. Prove that \( \prod_{i=1}^{n} a_i = (\prod_{i=1}^{n} a_i)^{2n} \) for every positive integer \( n \) [see (1.12)].

11. For any positive integer \( n \), prove in two different ways that
\[ \sum_{i=1}^{n} i(i!) = (n+1)! - 1. \]
   \textbf{Hint:} For one way, note that the first \( i \) of the expression being summed can be written as \((i+1) - 1\) and then see Exercise 4 of Section 1.1.

12. Let \( F_n \) denote the \( n \)th Fibonacci number and prove that the following are true for every positive integer \( n \).

a) \[ \sum_{i=1}^{n} F_i = F_{n+2} - 1 \]
b) \[ \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \]
c) \[ \sum_{i=1}^{n} F_{2i-1} = F_{2n} \]
d) \[ \sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1 \]

(e) \[ \sum_{i=1}^{n} (-1)^{i-1} F_i = (-1)^{n-1} F_{n-1} + 1 \]

13. Let \( \alpha = (1 + \sqrt{5})/2 \) and \( \beta = (1 - \sqrt{5})/2 \) so that \( \alpha \) and \( \beta \) are the roots of \( x^2 = x + 1 \); that is, \( \alpha^2 = \alpha + 1 \) and \( \beta^2 = \beta + 1 \). Prove that \( F_n = (\alpha^n - \beta^n)/\sqrt{5} \) for all \( n \geq 0 \).
   \textbf{Hint:} You may use either \( I_2 \) or \( I_4 \); in either case start by proving the result for \( n = 1 \) and \( n = 2 \). Why? This formula is due to J. P. M. Binet in 1843.

\textbf{Note:} \( F_0 = 0 \).