4. Prove that 
\[ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \]
for every positive integer \( n \).

5. Prove that 
\[ \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \]
for every positive integer \( n \).

6. Use \( I_1 \) to prove that \( 2^{2k} - 1 \) is divisible by 3 for every positive integer \( n \).
   \textbf{Hint:} For the second part of the proof make your assumption by assuming that there exists an integer \( q \) such that \( 2^{2k} - 1 = 3q \). Then consider 
\[ 2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 3 \cdot 2^{2k} + 2^{2k} - 1 = 3 \cdot 2^{2k} + 3q. \]

7. Prove that \( 2^{2n-1} + 1 \) is divisible by 3 for every positive integer \( n \).

8. Prove that \( f(n) = 3n^2 + 5n^3 + 7n \) is divisible by 15 for every integer \( n \).
   \textbf{Hint:} Note that \( f(-n) = -f(n) \).

9. Prove that \( 3^{2n+1} + 2^{n+2} \) is divisible by 7 for every nonnegative integer \( n \).

10. Prove that \( \Pi_{i=1}^{n} a_i = (\Pi_{i=1}^{n} a_i)^r \) for every positive integer \( n \) [see (1.12)].

11. For any positive integer \( n \), prove in two different ways that 
\[ \sum_{i=1}^{n} i(i!) = (n+1)! - 1. \]
   \textbf{Hint:} For one way, note that the first \( i \) of the expression being summed can be written as \((i+1) - 1 \) and then see Exercise 4 of Section 1.1.

12. Let \( F_n \) denote the \( n \)th Fibonacci number and prove that the following are true for every positive integer \( n \).

   (a) \[ \sum_{i=1}^{n} F_i = F_{n+2} - 1 \]

   (b) \[ \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \]

   (c) \[ \sum_{i=1}^{n} F_{2i-1} = F_{2n} \]

   (d) \[ \sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1 \]

   (e) \[ \sum_{i=1}^{n} (-1)^{i-1} F_i = (-1)^{n-1} F_{n-1} + 1 \]

13. Let \( \alpha = (1 + \sqrt{5})/2 \) and \( \beta = (1 - \sqrt{5})/2 \) so that \( \alpha \) and \( \beta \) are the roots of \( x^2 = x + 1 \); that is, \( \alpha^2 = \alpha + 1 \) and \( \beta^2 = \beta + 1 \). Prove that \( F_n = (\alpha^n - \beta^n)/\sqrt{5} \) for all \( n \geq 0 \).
   \textbf{Hint:} You may use either \( I_2 \) or \( I_4 \); in either case start by proving the result for \( n = 1 \) and \( n = 2 \). Why? This formula is due to J. P. M. Binet in 1843.

   \textbf{Note:} \( F_0 = 0 \).
$b < a < 2b$ and that $0 < a - b < b$. Thus, $a - b$ is a positive integer less than $b$. But since $\sqrt{2} = a/b$, it follows that

$$2 = \frac{a^2}{b^2},$$

$$2b^2 = a^2,$$

$$2b^2 - ab = a^2 - ab,$$

$$b(2b - a) = a(a - b),$$

$$\sqrt{2} = \frac{a}{b} = \frac{2b - a}{a - b}.$$

But this gives $\sqrt{2}$ as a ratio of two integers with positive integer denominator less than $b$. Since this is a contradiction, the theorem is true.

**EXERCISES 1.5**

1. Use the well-ordering principle to prove that $\sqrt{5}$ is irrational.

2. The Archimedean axiom states that if $a$ and $b$ are positive integers, there exists an integer $n$ such that $an \geq b$. Use the well-ordering principle to prove that this is so. *Hint:* Suppose that the assertion is false and consider the set $C$ of all positive integers of the form $b - ma$.

3. Use the well-ordering principle to prove that any nonempty set $C$ of integers none of which is less than a specified integer $a$ has a least element. *Hint:* Consider the set $D$ of all integers of the form $c - a + 1$, where $c$ is an element of $C$.

4. Use the well-ordering principle (as modified in Exercise 3 with $a = 2$) to prove that every integer $n \geq 2$ is either a prime or a product of primes.

5. Use Fermat's method of descent to prove that $\sum_{i=1}^{n} i = n(n + 1)/2$. Note that the critical arithmetic of the argument is essentially the same as in the proof of this result by $I_1$ in Section 1.4.

6. Use Fermat's method of descent to prove that

$$\sum_{i=1}^{n} i^3 = n^2(n + 1)^2/4.$$

**Computer Exercise**

7. Write a program to determine the least positive integer that can be written (nontrivially) as the sum of two cubes of positive integers in two different ways.
integers with the desired properties. Suppose that \( a = bq' + r' \), where \( 0 \leq r' < b \). It suffices to show that \( r = r' \) and \( q = q' \). If \( q' < q \), then \( q' + 1 \leq q \) since \( q \) and \( q' \) are both integers. Therefore,

\[
r = a - bq \leq a - b(q' + 1) = a - bq' - b = r' - b < 0,
\]

and this is a contradiction. Similarly, we obtain a contradiction if \( q' > q \). Thus, it must be the case that \( q = q' \). But then \( bq + r = a = bq + r' \), so \( r = r' \) as well.

Stated somewhat differently, this theorem simply says that if one divides \( a \) by the positive integer \( b \), one obtains a quotient \( q \) and a remainder \( r \) where \( r \) is nonnegative and less than \( b \). However, the restriction that \( b \) be positive is not strictly necessary, and the theorem could also be written in the form: Given integers \( a \) and \( b \) with \( b \neq 0 \), there exist unique integers \( q \) and \( r \) with \( 0 \leq r < |b| \) such that \( a = bq + r \).

The division algorithm is surprisingly useful, as we shall see subsequently. As a first example, note that with \( b = 2 \), the theorem implies that every integer \( a \) is either of the form \( 2k \) or of the form \( 2k + 1 \) (i.e., even or odd). Thus, \( a^2 \) is either of the form \( 4k^2 = 4r \) or \( 4k^2 + 4k + 1 = 4s + 1 \). Hence, the square of an integer must leave a remainder of 0 or 1 when divided by 4; it cannot leave a remainder of 2 or 3. Similarly, any integer \( a \) must be of the form \( 3k \), or \( 3k + 1 \), or \( 3k + 2 \). Thus, \( a^2 \) must be of the form \( 9k^2 = 3u \), or \( 9k^2 + 6k + 1 = 3v + 1 \), or \( 9k^2 + 12k + 4 = 3w + 1 \). Hence, the square of an integer must leave a remainder of 0 or 1 when divided by 3; it cannot leave a remainder of 2. Admittedly, these are only small results, but they are not without interest and they indicate an important way in which the division algorithm can be used.

**EXERCISES 1.7**

1. Prove that no number in the sequence 11, 111, 1111, 11111, \ldots , is a perfect square.
2. If \( p \) is a prime other than 2 or 5, prove that \( p \) must be one of the forms \( 10k + 1 \), \( 10k + 3 \), \( 10k + 7 \), or \( 10k + 9 \).
3. Prove that the product of any two odd numbers must be odd.
4. Prove that one of any two consecutive integers must be even.
5. Prove that one of any three consecutive integers must be divisible by 3.
6. If \( a \) is an integer, prove that one of the numbers \( a, a + 2, \) and \( a + 4 \) is divisible by 3.
7. If \( n \) is an integer not divisible by 2 or 3, show that \( n^2 + 23 \) must be divisible by 24. *Hint:* Any integer must be of the form \( 6k, 6k + 1, \ldots , \) or \( 6k + 5 \).
8. If \( a, b, \) and \( c \) are integers with \( a^2 + b^2 = c^2 \), show that \( a \) and \( b \) cannot both be odd.
9. If \( a \) and \( b \) are integers with \( b < 0 \), prove that there exist unique integers \( q \) and \( r \) with \( 0 \leq r < |b| \) such that \( a = bq + r \).