§ 1.4:

13. \( n = 1 \):
   \[ F_1 = 1 = \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} = \frac{\alpha - \beta}{\sqrt{5}} \]

   \( n = 2 \):
   \[ \frac{\alpha^2 - \beta^2}{\sqrt{5}} = \frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5)}{4\sqrt{5}} = \frac{1}{2} = F_2 \]

   Assume the formula holds true for \( n \), \( 1 \leq n \leq k \) with \( k \geq 2 \).

   Then
   \[ F_{k+1} = F_k + F_{k-1} = \frac{\alpha^k - \beta^k}{\sqrt{5}} + \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}} \]
   \[ = \frac{\alpha^{k-1}(1 + \alpha) - \beta^{k-1}(1 + \beta)}{\sqrt{5}} \]
   \[ = \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}} \]

   It holds for \( n = k+1 \). Thus by the principle of the Math Induction,
   we prove it for all positive integers.

§ 1.5:

1. Assume \( \sqrt{5} \) is rational, say \( \sqrt{5} = \frac{a}{b} \) with \( b \) the least positive integers among such representations.
   Then \( 2 < \frac{a}{b} < 3 \). Moreover,
   \[ 5b^2 = a^2 \]
   \[ 5b^2 - 2ab = a^2 - 2ab \]
   \[ b(5b - 2a) = a(0 - 2b) \]
   \[ \frac{a}{b} = \frac{5b - 2a}{a - 2b} \]
But $0 < a - 2b < b$. This contradicts with the assumption that $b$ is the least positive integer among all such denominators. Therefore $\sqrt{5}$ is irrational.

§1.7:

1. Numbers of the form \( \underbrace{11 \ldots 1}_{n\text{ digits}} = 10^{n-1} + 10^{n-2} + \ldots + 10 + 1 (n \geq 2) \)

must be of the form $4k + 3$ for some $k$.

But by discussion in class, odd perfect squares must be of the form $4k + 1$ (actually $8k + 1$ to be more precise).

Hence $\underbrace{11 \ldots 1}_{n\text{ digits}}$ cannot be a perfect square.

Method proved by Ex. 5 in §1.7.

6. (1) $a, a + 1, a + 2$ are three consecutive integers, so one of them is divisible by 3. If $a$ or $a + 2$ is divisible by 3, then we are done. If $a + 1$ is divisible by 3, then so is $a + 4 = a + 1 + 3$. Anyway, we are done.

(remark. There are multiple ways to prove this. Say by induction or division algorithm.)
Method

2. prove by division alg

\[ a = 3q + r, \quad 0 \leq r \leq 3 \]

If \( r = 0 \), then \( 3 | a \) \( (= 3q) \)
If \( r = 1 \), then \( 3 | a + 2 \) \( (= 3q+3) \)
If \( r = 2 \), then \( 3 | a + 4 \) \( (= 3q+6) \)

In any case, we are done.

By. We argue by contradiction. If both \( a \) and \( b \) are odd, then \( a^2 = 4m+1 \) and \( b^2 = 4n+1 \) for some integers \( m \) and \( n \). Then

\[ a^2 + b^2 = 4(m^2+2mn+n) + 2 = c^2 \]

Therefore \( c \) must be even, and \( 4 \mid c^2 \).

By Thm 2.1, property (v): \( 4 \mid 2 \) which is absurd!

Thus \( a \) and \( b \) cannot both be odd.