"Show work" for full credit problems

1 Evaluate $\int_C f$ where $f(z) = x^2 + 2yi$ and $C$ is given by $z(t) = t + 1 - ti$, $t \in [0, 1]$.

\[
\int_C f = \int_0^1 \left( (t+1)^2 + 2t^2i \right) (1-i) \, dt
\]

\[
= \int_0^1 \left( t^2 + 2ti + 1 - 2t^2i \right)(1-i) \, dt
\]

\[
= \left[ \left( \frac{t^3}{3} + t^2 + t - \frac{2t^2}{2} \right)i \right]_0^1
\]

\[
= (1 - i) \left( \frac{1}{3} + 1 + 1 - \frac{2}{2}i \right)
\]

\[
= (-i) \left( \frac{7}{3} - \frac{4}{3}i \right)
\]

\[
= \frac{7}{3} - \frac{4}{3}i - \left( \frac{7}{3} + \frac{4}{3}i \right) \frac{1}{1}
\]

\[
= \frac{4}{3} - \frac{10}{3}i
\]

2 Compute $\int_C e^z(z + 1)dz$ where $C$ is given by $z(t) = \cos t + i \sin t$, $t \in [0, \pi]$.

Notice that $(2,e^2)' = e^2 + 2e^2$

So

\[
\int_C e^z(z+1) \, dz = \left. e^z \right|_{2(\pi)}^{2(0)}
\]

\[
= e^0 \left|_{-1}^{1} \right.
\]

\[
= e - e^{-1}
\]
3 Let $f(z) = \frac{1}{4 - z^2}$.
(a) Find the power series expansion of $f$ around $z = 0$.

$$f(z) = \frac{1}{4} \frac{1}{1 - \left(\frac{z}{2}\right)^2} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{1}{2^{2n+2}} z^{2n}$$

When $|z| < 2$.

(b) Find the Laurent series expansion of $f$ around $z = 2$.

$$f(z) = \frac{1}{(2-z)(2+z)} = \frac{1}{4} \cdot \frac{1}{(2-z)(1 + \frac{2-z}{4})}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2-z}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{8}{4^{n+1}} (-1)^{n+1} (2-z)^{n-1}$$

When $0 < |2-z| < 4$

4 Find the radius of convergence of the power series expansion of $\frac{z+1}{z(z+1)}$ about $z = 2$. (Do not attempt to find the power series itself!)

The function has poles at $z = 0$ and $z = \pm 2i$, and is analytic everywhere else. By our theorem, the largest disk centered at 2 and contained in the domain has radius $|2-0| = 2$. (Note that both $i$ and $-i$ are further from 2 than 0.) So the answer to the question is $[2]$. 

\[ \begin{array}{c}
\text{Diagram showing the circle centered at 2 with radius 2.}
\end{array} \]
5 Determine the domain of convergence for the Laurent series \( \sum_{n=-\infty}^{-1} \frac{z^n}{n!} + \sum_{n=0}^{\infty} (2z)^n \).

By the ratio test,
\[
\sum_{n=-\infty}^{-1} \frac{z^n}{n!} \text{ converges when } |z| > 0.
\]

(actually, it converges to \( e^{\frac{1}{2}} - 1 \)).

And \( \sum_{n=0}^{\infty} (2z)^n \text{ converges when } |z| < \frac{1}{2} \).

So the domain of convergence is \( 0 < |z| < \frac{1}{2} \).

6 Find all the singularities of the function \( \frac{(2z-4)e^{(z-1)^2}}{z^2-4z} \) and identify their types.

\( z = 2 \) removable
\( z = 1 \) essential
\( z = 0 \) pole
\( z = -2 \) pole
7 Suppose that $f$ is an entire function and $|f(z)| \leq 2016 + |z|^{1/2}$. Then show that $f$ must be constant.

Expand $f$ to power series at 0

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

which converges for all $z \in \mathbb{C}$.

and

$$c_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}} \, dz$$

for any $R > 0$.

By M-L Lemma,

$$|c_k| \leq \frac{1}{2\pi} \frac{2016 + R^{1/2}}{R^{k+1}} \cdot 2\pi R = \frac{2016 + R^{1/2}}{R^k}$$

If $k \geq 1$, then

$$\lim_{R \to \infty} \frac{2016 + R^{1/2}}{R^k} = 0,$$

Therefore $c_k = 0$ for $k \geq 1$.

So $f(z) \equiv c_0$ is constant.

8 (For MATH 610 students only) Suppose that $C$ is the unit circle and $f$ is analytic on $C$. Is it true that $\int_C f = 0$? If yes, explain why; if no, give a counterexample.

No. take $f(z) = \frac{1}{2}$

which is analytic on the unit circle.

but

$$\int_C f = 2\pi i \left( \begin{array}{l} \text{assuming } C \text{ is oriented} \\ \text{counterclockwise} \\ \text{otherwise, the integral} \\ \text{in } -2\pi i \end{array} \right)$$

is not 0.