ELEVENTH ANNUAL

INTERMOUNTAIN COLLEGIATE MATH COMPETITION

NAME:

INSTRUCTIONS:

- Please put your name only on the cover sheet. Do NOT put identifying information on any other page.

- Do NOT write in the ID number box.

- Provide complete solutions to each problem. If possible, include proofs.

- By participating in this competition you agree to release your name and scores for publication and publicity.

- Have fun and good luck!
Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that the following two conditions hold:

(i) For all $n \in \mathbb{Z}$ we have $f(n)f(-n) = f(n^2)$.

(ii) For all $m, n \in \mathbb{Z}$ we have $f(m + n) = f(m) + f(n) + 2mn$.

We observe we can write (i) as $f(m+n) - f(m) = f(m) - f(n)$.

Denoting $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = f(n) - n^2$, we have $g(m+n) = g(m) + g(n)$ and from here it follows by induction that $g(n) = n g(1)$, for any $n \in \mathbb{Z}$.

Let $g(1) = k$. Then $f(n) = n^2 + kn$, $\forall n \in \mathbb{Z}$. Substituting in (i) we have $n^2 - k^2 = n^2 + kn^2$, $\forall n$ which is equivalent to $k^2 = -k$.

We obtain $k = 0$ or $k = -1$ and these lead to the solutions $f_1(n) = n^2$ and $f_2(n) = n^2 - n$. 
Consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = ||x| - 1| \). Find all solutions \( x \in \mathbb{R} \) to

\[
(f \circ f \circ \cdots \circ f)(x) = x
\]

\( n \) times

with \( n \) a positive integer. (Note: The answer may depend on \( n \)).

Since the LHS is non-negative, we obtain \( x \geq 0 \). We are then reduced to

the function \( f : [0, \infty) \to [0, \infty) \), \( f(x) = |x - 1| \).

We compute

\[
(f \circ f)(x) = \begin{cases} 
2 - x, & x \in [0, 1) \\
2, & x \in [1, 2) \\
x - 2, & x \geq 2 
\end{cases}
\]

\[
(f \circ f \circ f)(x) = \begin{cases} 
1 - x, & x \in [0, 1) \\
x - 1, & x \in [1, 2) \\
3 - x, & x \in [2, 3) \\
x - 3, & x \in [3, 4) \\
\vdots \\
2k - x, & x \in [2k - 1, 2k) \\
x - 2k, & x \geq 2k 
\end{cases}
\]

By induction we prove that

\[
(f \circ f \circ \cdots \circ f)(x) = \begin{cases} 
2k - x, & x \in [0, 1) \\
2k, & x \in [1, 2) \\
x - 2k, & x \in [2, 3) \\
4 - x, & x \in [3, 4) \\
2k - 1, & x \in [2k - 1, 2k) \\
x - 4, & x \in [2k, 2k + 1) \\
\vdots \\
x - 2k, & x \geq 2k 
\end{cases}
\]

Then the equation has the solutions

i) if \( n \) even: any \( x \in [0, 1] \)

ii) if \( n \) odd: \( x = \frac{1}{2} \)
Find all pairs of nonnegative integers \(x, y\) such that
\[
\sqrt{x^2 + y + 1} + \sqrt{y^2 + x + 4}
\]
is an integer.

If \(x, y \in \mathbb{Z}\) and \(\sqrt{x^2 + y + 1} + \sqrt{y^2 + x + 4} \in \mathbb{Z}\), then \(x^2 + y + 1\) and \(y^2 + x + 4\) are perfect squares.

Since \(x^2 \leq x^2 + y + 1\), it follows \((x+1)^2 \leq x^2 + y + 1\) or \(2x \leq y\).

Since \(y^2 \leq y^2 + x + 4\), it follows \((y+1)^2 \leq y^2 + x + 4\) or \(2y \leq x + 3\).

The region \(\begin{cases} 2x \leq y \\ 2y \leq x + 3 \\ x \geq 0 \\ y \geq 0 \end{cases}\) is represented by the triangle.

The only points of integer coordinates in this region are \((0, 0), (1, 2), (0, 1)\) and a simple verification shows that \(x = 0\) and \(x = 1\) are the only solutions of the problem.
The two tangent lines to a circle $C$ at points $P \neq Q$ intersect at a point $A$, and similarly the two tangent lines to $C$ at points $P' \neq Q'$ intersect at a point $A'$. If $A'$ is on the line generated by $PQ$, prove that $A$ is on the line generated by $P'Q'$.

We chose a system of coordinates in which the origin is the center of the circle and $A$ is on the horizontal axis.

Denote $\alpha = \measuredangle AOP = \measuredangle AQA$.

Then the coordinates of the points are $A(\cos \alpha, 0)$, $P'(\cos(\alpha + \beta), \sin(\alpha + \beta))$.

We know that in $\triangle O0A'$ we have $0A' = \frac{\cos \alpha}{\cos(\alpha + \beta)}$ so they the triangle $\triangle OP'A'$ we have $\cos \alpha = \frac{OP'}{OA'} = \frac{\cos(\alpha + \beta)}{\cos \alpha}$.

The slope of the line $AP'$ is $\frac{\sin(\alpha + \beta)}{\sec \alpha - \cos(\alpha + \beta)}$. The lines $AP'$ and $AQ'$ coincide if they have the same slope which corresponds to $\sin(\alpha + \beta - \delta) [\sec \alpha - \cos(\alpha + \beta - \delta)] = \sin(\alpha + \beta - \delta) [\sec \alpha - \cos(\alpha + \beta + \delta)] \iff \sin(\sin(\alpha + \beta - \delta) - \sin(\alpha + \beta - \delta)) = \sin(\delta + \beta + \delta) \cos(\alpha + \beta - \delta) - \sin(\alpha + \beta - \delta) \cos(\alpha + \beta + \delta)$

$2 \sin \delta \cos(\alpha + \beta)$

The two sides are equal, as desired.

End of proof.
Let \( A \subseteq \mathbb{R} \) be a finite, non-empty set of real numbers, and let \( f : A \rightarrow A \) be a function. Assume for every \( x, y \in A \) with \( x \neq y \), it happens that \( |f(x) - f(y)| < |x - y| \). Prove there exists some \( a \in A \) such that \( f(a) = a \).

There are \( a, b \in A \) such that \( |a - b| = \min_{x, y \in A} |x - y| \). Then \( |f(a) - f(b)| < |a - b| \)

implies \( f(a) = f(b) \), hence \( f(a) \in A \) is a strict inclusion, \( \text{card } f(A) < \text{card } A \). Denote \( A_1 = f(A) \) and \( A_n = f(A_{n-1}) \). We similarly prove that \( \text{card } f_1(A_1) < \text{card } A \) etc. to obtain a decreasing sequence \( A > A_1 > A_2 > \ldots \) of finite sets. Then there is an \( n \in \mathbb{N} \) such that \( A_n \) has only one element \( a \). Since \( f(A_n) \subseteq A_n \), we have \( f(a) = a \).

Solution 2 by Dhruv Rohatgi: Denote \( a \in A \) the element for which the function \( x \mapsto |x - f(a)| \) is minimum. We prove that \( f(a) = a \) by contradiction. Indeed if \( f(a) \neq a \) then \( |f(f(a)) - f(a)| < |f(a) - a| \), contradiction.
Find all polynomials with real coefficients $P(x) \in \mathbb{R}[x]$ satisfying:

$$(x + 1)^3 P(x - 1) - (x - 1)^3 P(x + 1) = 4(x^2 - 1)P(x).$$

Taking $x=1$, we see there is a polynomial $Q(x) \in \mathbb{R}[x]$ such that $P(x) = xQ(x)$. We substitute back in the identity and we get

$$(x+1)^3Q(x-1) - (x-1)^3Q(x+1) = 4(x^2-1)Q(x).$$

We divide by $x^2-1$:

$$(x+1)^2Q(x-1) - (x-1)^2Q(x+1) = 4xQ(x) \quad \text{or}$$

$$x^2[Q(x-1) - Q(x+1)] + 2x[Q(x-1) + Q(x+1)] + [Q(x-1) - Q(x+1)] = 4xQ(x).$$

Let $n = \deg Q$ and $a_n$ be the leading coefficient of $Q(x)$. Then the leading term of $Q(x-1) - Q(x+1)$ is $-2na_n x^{-1}$ and of $Q(x-1) + Q(x+1)$ is $2a_n x^n$. Identifying the leading terms of the two sides of the equation, we have

$$x^2(-2na_n x^{-1}) + 2x \cdot 2a_n x^n + 0 = 4x a_n x^n \quad \text{or}$$

$$-2na_n + 4a_n = 4a_n.$$

Hence $n = 0$, so $Q(x)$ is a constant polynomial.

In conclusion, $P(x) = ax$ for any $a \in \mathbb{R}$. 
Determine

\[
\lim_{n \to \infty} n^2 \left[ \left( 1 + \frac{1}{n+1} \right)^{n+1} - \left( 1 + \frac{1}{n} \right)^n \right].
\]

Let \( f(x) = \left( 1 + \frac{1}{x} \right)^x \) which is differentiable on \((0, \infty)\) with

\[
f'(x) = x \ln(1 + \frac{1}{x}) \left[ \ln(1 + \frac{1}{x}) + \frac{1}{x + 1} \right] = \left( 1 + \frac{1}{x} \right)^x \left[ \ln(1 + \frac{1}{x}) - \frac{1}{x + 1} \right]
\]

For any \( n \in \mathbb{N} \) using the mean value theorem for \( f(x) \) on the interval \([n, n+1] \) there exist \( c_n \in (n, n+1) \) such that \( f(n+1) - f(n) = f'(c_n) \)

The limit to evaluate becomes

\[
\lim_{n \to \infty} n^2 f'(c_n) = \lim_{n \to \infty} \frac{n^2}{c_n^2} c_n^2 f'(c_n)
\]

But \( \lim_{n \to \infty} c_n = \infty \) and \( \lim_{n \to \infty} \frac{n^2}{c_n^2} = 1 \) (since \( n < c_n < n+1 \)) so the problem is reduced to the computation of \( \lim_{t \to \infty} t^2 f'(t) = \)

\[
\lim_{t \to \infty} t^2 \frac{\ln(1 + \frac{1}{t}) - \frac{1}{1+t}}{t^2}.
\]

It is known that \( \lim_{t \to \infty} \left( 1 + \frac{1}{t} \right)^t = e \)

On the other side denoting \( t = u \to 0^+ \) we have to evaluate

\[
\lim_{u \to 0^+} \frac{\ln(1+u) - \frac{1}{u+1}}{u^2} = \lim_{u \to 0^+} \frac{1}{2u} = \frac{1}{2}
\]

Hence the limit to compute is \( e \).