Argue by contradiction

**Problem 1.** Prove that there is no function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[
f(x) + f(1 - x) = x
\]
for any real \( x \).

**Problem 2.** Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.

**Problem 3.** Prove there are infinitely many prime integers.

**Problem 4.** Let \( a, b, c \) be odd integers. Prove that the equation \( ax^2 + bx + c = 0 \) cannot have rational solutions.

**Problem 5.** (Homework) Prove there is no arithmetic progression which has \( \sqrt{2}, \sqrt{3}, \sqrt{5} \) among its terms.

**Problem 6.** Prove that if \( 2^n + 1 \) is a prime number, then the positive integer \( n \) is a power of 2.

**Problem 7.** Let \( n \) be a positive integer. Prove that one of the integer numbers \( n, n + 1, n + 2, \ldots, 2n - 1, 2n \) is a perfect square.

**Problem 8.** Let \( S \) be a set rational numbers that is closed under addition and multiplication (that is, whenever \( a, b \) are members of \( S \), so are \( a + b \) and \( ab \)), and having the property that for every rational number \( r \) exactly one of the following three statements is true:
\[
r \in S, \quad -r \in S, \quad 0 \in S.
\]

a) Prove that 0 does not belong to \( S \).
b) Prove that all positive integers belong to \( S \).
c) Prove that \( S \) is the set of all positive rational numbers.

**Problem 9.** Let \( a, b, c \) be rational numbers such that \( a + b \sqrt{2} + c \sqrt{4} = 0 \). Prove that \( a = b = c = 0 \).