Problem 1. Let \( a \) be a fixed real number. Prove that \( a^2 \geq x + 1 \) for any \( x \in \mathbb{R} \) if and only if \( a = e \).

Solution: If \( a = e \) then we study the behavior of the function \( f(x) = e^x - x - 1 \). The derivative \( f'(x) = e^x - 1 \) has negative sign on \((-\infty, 0)\) and positive sign on \((0, \infty)\), so the function has a minimum at \( x = 0 \). But \( f(x) \geq f(0) \) can be written \( e^x \geq x + 1 \).

Suppose now \( a^2 \geq x + 1 \) for any \( x \). Taking \( x = 0 \) we see \( a \geq 2 \). Consider the function \( f(x) = a^2 - x - 1 \). The derivative \( f'(x) = 2a \) has the root \( x_0 = -\ln a \) which is a minimum for the function \( f \). By hypothesis \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \), therefore \( f(x_0) \geq 0 \). But \( f(x_0) = \frac{(\ln a)(\ln a - a(\ln a - 1))}{a \ln a} \leq \frac{(\ln a - 1)(\ln a - a(\ln a - 1))}{a \ln a} \leq 0 \). Consequently \( \ln a = \ln a - 1 = 0 \), so \( a = e \).

Problem 2. Let \( a > 0 \) be a fixed real number. Prove that \( a^2 \geq x^a \) for all \( x > 0 \) if and only if \( a = e \).

Solution: The inequality is equivalent with \( f(x) \leq f(a) \), for any \( x > 0 \), where \( f(x) = \frac{\ln x}{x} \). Therefore \( x = a \) is a maximum for \( f \), but this function has a maximum only in \( x = e \).

Problem 3. Consider the sequence defined by \( \left(1 + \frac{1}{n}\right)^{n+x_n} = e \). Prove that \( x_n \) is decreasing towards the limit \( \frac{1}{2} \).

Solution: By hypothesis \( x_n = f(n) \), where \( f(x) = \frac{1}{\ln(x+1) - \ln x} - x \). We prove \( f \) is increasing on \([1, \infty)\). It suffices to prove \( f'(x) = \frac{1}{x(x+1)\ln^2(1+1/x)} - 1 \geq 0 \) or equivalently \( g(x) = \frac{1}{\sqrt{x(x+1)}} - \ln(x+1) + \ln x \geq 0 \).

Since \( g'(x) = \frac{1}{x(x+1)} \left(1 - \frac{x+1/2}{\sqrt{x(x+1)}}\right) < 0 \), we have \( g(x) \geq \lim_{x \to \infty} g(x) = 0 \).

For the limit of the sequence it suffices to prove that \( \lim_{x \to \infty} f(x) = \frac{1}{2} \), or with \( y = \frac{1}{x} \), \( \lim_{y \to 0} \frac{1}{\ln(1+y)} - \frac{1}{y} = \frac{1}{2} \), which can be proved easily using l’Hôpital’s rule or a Taylor expansion.

Problem 4. Solve the equation \( 2^x + 2\sqrt{1-x^2} = 3 \).

Solution: Obviously, the solutions of the equation satisfy \( x \in [0, 1] \). We prove \( x = 0 \) and \( x = 1 \) are the only solutions of the equation. The function \( f : [0, 1] \to \mathbb{R} \) is increasing on the interval \([0, \sqrt{2}/2]\) and decreasing on \([\sqrt{2}/2, 1]\). The identity \( f(x) = f(\sqrt{1-x^2}) \) makes it sufficient to prove \( f \) increasing on \([0, \sqrt{2}/2]\). The function \( g(x) = \frac{2x}{x^2} \) is decreasing on \([0, 1]\), since \( g'(x) = \frac{2x}{x^2} (x \ln 2 - 1) < 0 \). Then \( f'(x) = x \ln 2 \left(g(x) - g(\sqrt{1-x^2})\right) > 0 \), since \( x < \sqrt{1-x^2} \) on \([0, \sqrt{2}/2]\).

Problem 5. Let \( f(x) \) be a positive-valued function over the reals such that \( f'(x) > f(x) \) for all \( x \). For what \( k \) must there exist \( N \) such that \( f(x) > e^{kx} \) for \( x > N \)? [P1994]

Solution: The condition can be written \( g'(x) > 0 \), where \( g(x) = \ln f(x) - x \) and an equivalent form for \( f(x) > e^{kx} \) is

\[
(0.1) \quad g(x) > (k-1)x
\]

For \( k \geq 1 \), and \( g(x) = \frac{k-1}{2}x \) there is no \( N \) such that \( 0.1 \) is satisfied for \( x > N \). If \( k < 1 \) then \( \lim_{x \to \infty} g(x) - (k-1)x = \infty \), so there is an \( N \) such that for \( x > N \) \( 0.1 \) holds.

Problem 6. Let \( f \) be an infinitely differentiable real-valued function defined on the real numbers. If

\[
f\left(\frac{1}{n}\right) = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \ldots,
\]

compute the values of the derivatives \( f^{(k)}(0), k = 1, 2, 3, \ldots \). [P1992]

Solution: With \( g(x) = f(x) - \frac{1}{1 + x^2} \) we have \( g\left(\frac{1}{n}\right) = 0, \quad n = 1, 2, 3, \ldots \). By the theorem of Rolle there is a sequence \( (x^{(1)}_n)_n \) such that \( \frac{1}{n+1} < x^{(1)}_n < \frac{1}{n} \) and \( g'(x^{(1)}_n) = 0 \). As a consequence of \((*)\) the sequence \( x^{(1)}_n \) is decreasing
to 0, hence \( g'(0) = 0 \). Using again Rolle’s theorem there is a sequence \((x_n^{(2)})\) such that \( x_{n+1}^{(1)} < x_n^{(2)} < x_n^{(1)} \) (**) and \( g^{(2)}(x_n^{(2)}) = 0 \). From (**), \( x_n^{(2)} \) is decreasing to 0, and \( g^{(2)}(0) = 0 \). By recurrence we prove in this way that \( g^{(k)}(0) = 0 \), for any \( k \). Thus \( f^{(k)}(0) = \phi^{(k)}(0) \), where \( \phi(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots \) and consequently \( f^{(2k+1)}(0) = 0 \), \( f^{(2k)}(0) = (-1)^n(2n)! \).

**Problem 7.** Let \( f \) be a function such that \( f(1) = 1 \) and \( f'(x) = \frac{1}{x^2 + f^2(x)} \) for all \( x \geq 1 \). Show that \( \lim_{x \to \infty} f(x) \) exists and is less than \( 1 + \frac{\pi}{4} \).

**Solution:** The function has positive derivative so is increasing. Then the limit \( \lim_{x \to \infty} f(x) \) exists and \( f'(x) \leq \frac{1}{x^2 + f(1)^2} = \frac{1}{x^2 + 1} \) for any \( x \geq 1 \), which proves that the function \( g(x) = f(x) - \arctan x \) is decreasing. Hence \( g(x) \leq g(1) = 1 - \frac{\pi}{4} \) and consequently \( f(x) \leq \arctan x + 1 - \frac{\pi}{4} \). Therefore \( \lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} \left( \arctan x + 1 - \frac{\pi}{4} \right) = 1 + \frac{\pi}{4} \).

**Problem 8.** Let \( f : (-2, 2) \to \mathbb{R} \) be a function of class \( C^2 \) such that \( f(0) = 0 \). Show that the sequence \((u_n)\) defined by \( u_n = \sum_{k=1}^{n} f\left(\frac{x}{n}\right) \) is convergent and compute its limit.

**Solution:** Let \( c_k \in \left(0, \frac{k}{n^2}\right)\) be such that \( f\left(\frac{k}{n^2}\right) - f(0) = \frac{k}{n^2} f'(c_k) \), and \( d_k \in (0, c_k)\) such that \( f'(c_k) - f'(0) = c_k f''(d_k) \). Then \( |u_n| = \sum_{k=1}^{n} \frac{k}{n^2} f'(0) = |\sum_{k=1}^{n} \frac{k}{n^2} f''(d_k)| \leq \sum_{k=1}^{n} \frac{k^2}{n^2} \max_{x \in [0, 1]} |f''(x)| \). Passing at limit with \( n \to \infty \) we obtain \( \lim_{n \to \infty} u_n = f'(0) \).

**Problem 9.** a) Solve in real numbers the equation \( 2x + 4x^2 = 3x + 5x^3 \).

b) Solve in real numbers the equation \( 2x + 5x^2 = 3x + 4x^3 \).

**Solution:** a) For \( x > 0 \), \( 3x + 5x^2 > 2x + 4x^2 \), and for \( x < 0 \), \( 3x + 5x^2 < 2x + 4x^2 \), so \( x = 0 \) is the only solution.

b) The mean value theorem for the function \( f(t) = t^x \) shows there is \( c \in (2, 3) \) and \( d \in (4, 5) \) such that \( 3x - 2x = xc^{x-1} \) and \( 5x - 4x = xd^{x-1} \). The equation becomes \( x \left[ \left( \frac{d}{c} \right)^{x-1} - 1 \right] = 0 \), and has the solutions \( x = 0 \) and \( x = 1 \).

**Problem 10.** Solve in real numbers the equation \( 5x + 5x^2 = 4x + 6x^2 \).

**Solution:** Write the equation as \( 5x - 4x = 6x^2 - 5x^2 \). Using the mean theorem for the function \( f(t) = t^x \), there is \( c \in (4, 5) \) such that \( 5x - 4x = xc^{x-1} \). Similarly for the function \( g(t) = t^x \), there is \( d \in (5, 6) \) such that \( 6x^2 - 5x^2 = x^2d^{x-1} \). The equation becomes \( xc^{x-1} = x^2d^{x-1} \). One obvious solution is \( x = 0 \). Looking for non-zero solutions, we see that necessarily \( x > 0 \).

Suppose \( 0 < x < 1 \). Then \( x = \left( \frac{d^{1+x}}{c} \right)^{1-x} > 1 \). Contradiction.

For \( x > 1 \), we obtain the contradiction \( x = \left( \frac{d^{1+x}}{c} \right)^{1-x} < 1 \). Therefore \( x = 0 \) and \( x = 1 \) are the only solutions.

**Problem 11.** Evaluate the limit \( \lim_{n \to \infty} n^2 \left( \left( 1 + \frac{1}{n+1} \right)^{n+1} - \left( 1 + \frac{1}{n} \right)^n \right) \).

**Problem 12.** Let \( f(x) = (x^2 - 1)^n \), and let \( P_n(x) \) be the \( n \)-th derivative of \( f(x) \). Prove that \( P_n(x) \) is a polynomial of degree \( n \) with \( n \) real, distinct roots in \((-1, 1)\).