Problem 1. Evaluate \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}} \).

Problem 2. Evaluate \( \lim_{n \to \infty} \frac{1}{n} \prod_{k=1}^{2n} (n^2 + k^2)^{1/n} \).

Problem 3. Let \( f : [0,1] \to \mathbb{R} \) be a continuous function. Show that
\[
\lim_{n \to \infty} \frac{1}{n} \int_{0}^{1} f(x)[nx] dx = \int_{0}^{1} xf(x) dx
\]

Problem 4. Let \( f \) be a differentiable function defined in the closed interval \([0,1]\) and such that \( |f'(x)| \leq M \) for any \( x \in (0,1) \). Prove that
\[
\left| \int_{0}^{1} f(x) dx - \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) \right| \leq \frac{M}{n}
\]

Problem 5. Evaluate \( \lim_{n \to \infty} n \int_{0}^{1} \frac{x^{2n}}{x+1} dx \).

Problem 6. Let \( f \) be a function such that \( f' \) is continuous. Evaluate
\[
\lim_{n \to \infty} \int_{0}^{1} f(x) \sin nx dx
\]

Problem 7. If \( f \) is an invertible function and \( f' \) is continuous, prove that
\[
\int_{a}^{b} f(x) dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y) dy
\]
In the case where \( f \) and \( g \) are positive functions and \( b > a > 0 \), draw a diagram to give a geometric interpretation of this identity.

Problem 8. Evaluate \( I = \int_{0}^{1} (\sqrt{1-x^2} - \sqrt{1-x^3}) dx \).

Problem 9. Find \( \lim_{h \to 0} \frac{1}{h} \int_{2}^{2+h} \sqrt{1+t^4} dt \).

Problem 10. Suppose \( f \) is differentiable on \([0,1]\), \( f(0) = 0 \), \( f(1) = 1 \), \( f'(x) > 0 \), and \( \int_{0}^{1} f(x) dx = 1/3 \). Find the value of the integral \( \int_{0}^{1} f^{-1}(y) dy \) and give a geometric interpretation of the result.

Problem 11. Evaluate \( \int_{a}^{b} |x| dx \), where \( a \) and \( b \) are real numbers with \( 0 \leq a < b \).

Problem 12. Find the interval \([a,b]\) for which the value of the integral \( \int_{a}^{b} (2+x-x^2) dx \) is a maximum.

Problem 13. Suppose \( f \) is continuous on the interval \([-a,a]\).

a) If \( f \) is even (i.e \( f(-x) = f(x) \)), then \( \int_{-a}^{a} f(x) dx = 2 \int_{0}^{a} f(x) dx \). Interpret this equality on a figure.

b) If \( f \) is odd (i.e. \( f(-x) = -f(x) \)), then \( \int_{-a}^{a} f(x) dx = 0 \). Interpret this equality on a figure.

c) Evaluate \( \int_{-\pi/2}^{\pi/2} x^2 \sin x dx \).

Problem 14. Evaluate the limit \( \lim_{y \to 0} \frac{1}{y} \int_{0}^{\pi} \tan y \sin x dx \).

Problem 15. Define \( C(\alpha) \) to be the coefficient of \( x^{1992} \) in the power series about \( x = 0 \) of \((1+x)^\alpha\). Evaluate
\[
\int_{0}^{1} \left( C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.
\]

[P1992]
Problem 16. Show that the function \( F : [0, 1) \to \mathbb{R} \), defined by \( F(t) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - t \cos^2 x}} \) is an increasing function.

Problem 17. Determine the value of the integral \( I_n = \int_0^\pi \frac{\sin^2 nx}{\sin^2 x} \, dx \).

Problem 18. Evaluate \( \int_0^\pi \frac{dx}{x + x^{10}} \).

Problem 19. Let \( F(x) \) be a differential function such that \( F'(a - x) = F'(x) \) for all \( x \in [0, a] \). Evaluate \( I = \int_0^a F(x) \, dx \) and give an example of such a function \( F(x) \).

Problem 20. Let \( f(x) \) be a continuous function on \([0, a]\), where \( a > 0 \), such that \( f(x)f(a - x) = 1 \). Prove that there are infinitely many such functions, and evaluate the integral \( I = \int_0^a \frac{dx}{1 + f(x)} \).

Solution: For any \( c > 0 \), the function \( f(x) = e^{c-x^2} \) satisfies the condition \( f(x)f(a - x) = 1 \). The substitution \( y = a - x \) yields, \( I = \int_0^a \frac{dy}{1 + f(a - y)} = \int_0^a \frac{f(x) \, dx}{1 + f(x)} \). Therefore \( I + I = \int_0^a \frac{dx}{1 + f(x)} + \int_0^a \frac{f(x) \, dx}{1 + f(x)} = a \) and \( I = \frac{a}{2} \).

Problem 21. Evaluate
\[
\int_2^4 \frac{\sqrt{\ln(9 - x)} \, dx}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}}.
\]

[P1987]

Problem 22. Evaluate \( \int_0^\frac{\pi}{2} \sin x \, dx \)

Problem 23. Let \( f(x) \) be a continuous function on \([0, a]\), where \( a > 0 \), such that \( f(x) + f(a - x) \) does not vanish on \([0, a]\). Evaluate the integral \( I = \int_0^a f(x) \, dx \).

Problem 24. Evaluate \( \int_0^\frac{\pi}{2} \frac{dx}{1 + \tan x} \).

Problem 25. Let \( I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) \, dx \). For which integers \( m \), \( 1 \leq m \leq 10 \) is \( I_m \neq 0 \)? [P1985]

Problem 26. Let \( 0 < a < b \). Evaluate \( I = \lim_{t \to 0} \left\{ \int_0^1 (bx + (a - 1)x) \, dx \right\}^{1/t} \).

Problem 27. Show that for all \( x > 0 \),
\[
0 < \int_0^\infty \frac{\sin t}{\ln(1 + x + t)} \, dt < \frac{2}{\ln(1 + x)}
\]

Problem 28. Let \( p > 0 \) be a real number and let \( n \geq 0 \) be an integer. Evaluate
\[
u_n(p) = \int_0^\infty e^{-px} \sin xd\]

Problem 29. For what pairs \((a, b)\) of positive real numbers does the improper integral
\[
\int_{x=b}^\infty \left( \sqrt{x + a} - \sqrt{x} - \sqrt{x - b} \right) \, dx
\]
converge? [P1995]

Problem 30. Study the convergence of the integral \( f(h) = \int_0^\infty e^{-hx} \sin x \, dx \).

Problem 31. Evaluate \( \int_0^\infty \frac{\arctan(\pi x)}{x} \, dx \).
Problem 32. Find the value of the constant $C$ for which the integral
\[
\int_0^\infty \left( \frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) \, dx
\]
converges.

Problem 33. Evaluate $J = \int_0^{\pi/2} \ln(x) \, dx$.

Problem 34. Find all positive $a$ such that $\int_0^a x^{-\ln x} \, dx = \int_a^\infty x^{-\ln x} \, dx$. Evaluate the integrals for these values of $a$.

Problem 35. Evaluate
\[
\int_0^\infty \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) \, dx.
\]

Problem 36. Given that $\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$ evaluate $\int_0^\infty \frac{\sin^2 x}{x^2} \, dx$.

Problem 37. The base of a solid is a square with vertices located at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Each cross section perpendicular to the $x$-axis is a semi-circle. Find the volume of the solid.

Problem 38. By comparing areas, show that $\ln 2 < 1 < \ln 3$. Deduce that $2 < e < 3$.

Problem 39. By comparing areas, show that
\[
\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}
\]

Problem 40. Let the function $f$ from $[0,1]$ to $[0,1]$ have the following properties:
- $f$ is of class $C^1$,
- $f(0) = f(1) = 0$,
- $f$ is nonincreasing (i.e. $f$ is concave).

Prove that the arclength of the graph of $f$ does not exceed 3.

Problem 41. Can an arc of a parabola inside a circle of radius 1 have a length greater than 4? [P2001]

Problem 42. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find $c$ so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the $y$-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.] [P1993]

Problem 43. Find the positive value of $m$ such that the area in the first quadrant enclosed by the ellipse $\frac{x^2}{4} + y^2 = 1$, the $x$-axis, and the line $y = 2x/3$ is equal to the area in the first quadrant enclosed by the ellipse $\frac{x^2}{4} + y^2 = 1$, the $y$-axis, and the line $y = mx$. [P1994]

Problem 44. Prove or disprove: There is a region $R$ in the plane which has infinite area and such that the volume of the solid $S$ obtained by rotating $R$ about the $x$-axis is finite, and the surface area of $S$ is infinite.

Problem 45. Show that the surface area of a zone of the sphere that lies between two parallel planes is $S = \pi dh$, where $d$ is the diameter and $h$ the distance between the planes.

Problem 46. For what values of $m$ do the line $y = mx$ and the curve $y = f(x) = \frac{x}{x^2 + 1}$ enclose a region? Find the area of the region.

Problem 47. Find the area of the region consisting of all points inside a square that are closer to the center than to the sides of the square.

Problem 48. (Theorem of Pappus) Let $R$ be a region that lies entirely on one side of a line $l$ in the plane. If $R$ is rotated about $l$, then the volume of the resulting solid is the product of the area $A$ of $R$ and the distance $d$ traveled by the centroid of $R$.

Problem 49. A torus is formed by rotating a circle of radius $r$ about a line in the plane of the circle that is a distance $R > r$ from the center of the circle. Find the volume and the surface area of the torus.