UNR 2006 - Putnam exam preparation. Trigonometry

Problem 1. Solve the equation
\[ \sum_{k=1}^{n} \frac{1}{\cos x + \cos(2k+1)x} = \sum_{k=1}^{n} \frac{1}{\cos x - \cos(2k+1)x} \]

Problem 2. Let \( n \geq 2 \) be an integer. Solve the equation \( \sin^n x - \cos^n x = 1 \).

Problem 3. Consider the family of functions of real variable
\[ f_m(x) = \sqrt{\cos^4 x + m\sin^2 x} + \sqrt{\sin^4 x + m\cos^4 x} \]
Find the values of \( m \) for which \( f_m \) is a constant function.

Problem 4. For any \( b, c \) real numbers, prove that \( \cos(b + c) + \sqrt{2}(\sin b + \sin c) \leq 2 \).

Problem 5. Let \( x, y, z \) be real numbers different from \( \pm \frac{1}{\sqrt{3}} \), such that \( x + y + z = xyz \). Prove that
\[ \frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \frac{3y - y^3}{1 - 3y^2} \frac{3z - z^3}{1 - 3z^2} \]

Problem 6. If \( A \) and \( B \) are non-negative numbers, then
\[ \arctan A - \arctan B = \arctan \frac{A - B}{1 + AB} \]

Problem 7. Prove that \( \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2} \).

Problem 8. Let \( n \geq 2 \). Prove that
\[ \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}. \]