2.1: Rates of Change and Tangents to Curves

The *average rate of change* of the function \( y = f(x) \) on \([x_1, x_2]\) is

\[
m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}
\]

where \( h = \Delta x = x_2 - x_1 \).

**Examples:** In each case find the average rate of change of the function on the given interval:

1. \( g(x) = x^2 \), \([-1, 3]\), \( (m = 2) \)

2. \( h(x) = \cos x \), \([\pi/2, 2\pi]\), \( (m = -\frac{2}{3\pi}) \)

Note that \( m \) is the slope of the secant connecting \( P \) and \( Q \).
Average Velocity vs. Instantaneous Velocity

If \( y = f(t) \) is position as a function of time \( t \), then the average rate of change \( \frac{\Delta y}{\Delta t} \) may be interpreted as average velocity.

Consider \( f(t) = t^2 + 2t \) over the interval \([1, 1 + h]\).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( 1 + h )</th>
<th>( \Delta y/\Delta t )</th>
<th>( h )</th>
<th>( 1 + h )</th>
<th>( \Delta y/\Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
<td>-1</td>
<td>0</td>
<td>3</td>
</tr>
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<td>1.1</td>
<td>4.1</td>
<td>-0.1</td>
<td>0.9</td>
<td>3.9</td>
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<td>4.01</td>
<td>-0.01</td>
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</tr>
</tbody>
</table>

Note that in the above table it appears that the average velocity approaches a limit, namely 4, as \( h \) goes to 0. Hence, the instantaneous velocity at \( t_0 = 1 \) is \( v = 4 \).
Slope of Curve = Slope of Tangent

What happens to the slope of the secant as \( Q \) approaches \( P \) in the graph \( y = f(x) \)? As \( Q \) goes to \( P \), the slope of the secant approaches the slope of the tangent, or the slope of the curve, at the point \( P \). The slope of the tangent may be interpreted as the rate of change of \( y \) with respect to \( x \).

**Example:** Let’s find the slope of the curve \( y = x^2 + 2x \) at the point \( P(1, 3) \). Take \( x_1 = 1 \) and \( x_2 = 1 + h \). Then \( f(x_1) = 3 \),

\[
f(x_2) = f(1 + h) = (1 + h)^2 + 2(1 + h) = 3 + 4h + h^2
\]

Note that \( Q(1 + h, 3 + 4h + h^2) \). The slope of the secant is:

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{3 + 4h + h^2 - 3}{1 + h - 1} = 4 + h.
\]

So as \( Q \) goes to \( P \), the slope of the secant approaches 4, the slope of the curve at \( P \). Tangent line equation? \( y = 4x - 1 \).

See the widget on p. 79 for some graphical examples of this phenomenon.