2.3: Calculating Limits using the Limit Laws

Some easy limits ($k$ is a constant):

\[
\begin{align*}
\lim_{x \to c} x &= c \\
\lim_{x \to c} |x| &= |c| \\
\lim_{x \to c} k &= k \\
\lim_{x \to c} kf(x) &= k \lim_{x \to c} f(x).
\end{align*}
\]

Suppose that \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \)

- **Sum Rule**: \( \lim_{x \to c} (f(x) \pm g(x)) = L \pm M \)
- **Product Rule**: \( \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M \)
- **Quotient Rule**: \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \) if \( M \neq 0 \)
- **Power Rule**: \( \lim_{x \to c} (f(x))^n = L^n \) where \( n \) is a positive integer
- **Root Rule**: \( \lim_{x \to c} (f(x))^{1/n} = L^{1/n} \) (need \( L \geq 0 \) if \( n \) even)
- **Polynomial Rule**: \( \lim_{x \to c} p(x) = p(c) \) if \( p(x) \) is a polynomial

Note if \( f(x) = \frac{p(x)}{q(x)} \) with \( p(x) \) and \( q(x) \) polynomials (that is, \( f(x) \) is a rational function), then \( \lim_{x \to c} f(x) = f(c) \) if \( q(c) \neq 0 \).
1. \( \lim_{x \to 3} 2x - 1 = 5 \)

2. \( \lim_{x \to 2} \sqrt{3x^2 - 2x - 1} = \sqrt{7} \) (use polynomial and root rules)

3. \( \lim_{x \to 3} \frac{x^2 + x - 2}{x - 1} = 5 \) (use polynomial and quotient rules)

4. \( \lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3 \) (this is a rational function)

In Example 4, observe that \( \lim_{x \to 1} x - 1 = 0 \) so we can’t use the quotient rule. But \( \lim_{x \to 1} x^2 + x - 2 = 0 \) also. Let’s try factoring:

\[
\frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2 \quad \text{if } x \neq 1.
\]

The limit does not depend on the value of the function at 1, so

\[
\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} x + 2 = 3.
\]
Now show that

1. \( \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = -4 \)

2. \( \lim_{x \to -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3} = \frac{3}{2} \)

3. \( \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = -\frac{1}{4} \)

Examples of the form “0/0” can be tricky. Sometimes factoring does not work.

The **Sandwich Theorem** might help.

Suppose \( f(x) \leq g(x) \leq h(x) \) and

\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.
\]

Then

\[
\lim_{x \to a} g(x) = L.
\]

One can show that \(-|x| \leq \sin x \leq |x|\) for all \( x \). Since \( \lim_{x \to 0} |x| = \lim_{x \to 0} -|x| = 0 \) we have \( \lim_{x \to 0} \sin x = 0 \). Similarly we have \( \lim_{x \to 0} \cos x = 1 \) (use \(-|x| \leq 1 - \cos x \leq |x|\)).
A piecewise defined function

Let \( f \) be given below:

\[
    f(x) = \begin{cases} 
        \sqrt{1 - x^2} & \text{if } 0 \leq x < 1, \\
        1 & \text{if } 1 \leq x < 2, \\
        x^2 - 4x + 5 & \text{if } 2 < x \leq 3, \\
        8 - 2x & \text{if } 3 < x < 4 
    \end{cases}
\]

For \( a = 1, 2, 3 \) find the limit \( \lim_{x \to a} f(x) \), if it exists and then sketch a graph of \( f \).

We have

\[
    \lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1.
\]

Since \( 0 \neq 1 \), \( \lim_{x \to 1} f(x) \) does not exist.

We have \( \lim_{x \to 2^-} f(x) = 1 = \lim_{x \to 2^+} f(x) \).

Since the one-sided limits agree, \( \lim_{x \to 2} f(x) \) exists and \( \lim_{x \to 2} f(x) = 1 \).

We have \( \lim_{x \to 3^-} f(x) = 2 = \lim_{x \to 3^+} f(x) \).

Since the one-sided limits agree, \( \lim_{x \to 3} f(x) \) exists and \( \lim_{x \to 3} f(x) = 2 \).