2.5: Continuity

Definition

\( f(x) \) is continuous at an interior point \( c \) if

i. \( f(c) \) exists,

ii. \( \lim_{{x \to c}} f(x) \) exists,

iii. \( f(c) = \lim_{{x \to c}} f(x) \)

At an endpoint \( c \) use a 1-sided limit instead (say \( f(x) \) is right or left continuous at \( c \)).

Note \( \sqrt{x} \) is continuous at every \( c > 0 \) and (right) continuous at 0.

\[ y = \sqrt{x} \]

\( f(x) \) is (right) continuous at 0, discontinuous at 1, 2 and 4, and continuous at all other \( c \) in \([0, 4] \).

At each \( n \), \( \lfloor x \rfloor \) is right continuous but not left continuous (hence discontinuous).
\( f(x) \) is continuous on an interval if it is continuous at every point.

Every polynomial is continuous on \((-\infty, \infty)\), \(\sqrt{x} \) is continuous on \([0, \infty)\) and \(\sqrt{2x - x^2} \) is continuous on \([0, 2]\).

Every trig function is continuous at each point of its domain.

There are many ways for a function to be discontinuous at \(c\).

i \( \lim_{x \to c} f(x) \) exists, but either
   a \( f(c) \) is not defined, or
   b \( f(c) \neq \lim_{x \to c} f(x) \).

This is called a **removable discontinuity**.

ii \( \lim_{x \to c} f(x) \) does not exist.
   a Both 1-sided limits exist but \( \lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x) \), called a **jump discontinuity**.
   b \( \lim_{x \to c^+} f(x) = \pm \infty \) or \( \lim_{x \to c^-} f(x) = \pm \infty \), called a **infinite discontinuity**.
   c Other wild behavior including **oscillating discontinuities**.
All graphs except (a) represent discontinuities at 0: Removable discontinuities in (b) and (c); jump discontinuity in (c); an infinite discontinuity in (e) and an oscillating discontinuity in (f).
If \( f(x) \) and \( g(x) \) are both continuous at \( c \) then our limit laws guarantee that 
\[ f(x) \pm g(x), \; f(x) \cdot g(x) \text{ and } \frac{f(x)}{g(x)} \text{ (if } g(c) \neq 0) \] are also continuous at \( c \); similar results hold for powers and roots. Hence, a 
rationa function is continuous at each point of its domain.

- Is \( f(x) \) continuous on \(( -\infty, \infty )\)?

\[
f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}
\]

- How about \( g(x) \)?

\[
g(x) = \frac{x^2 - 2x + 3}{x^2 + 3x + 5}.
\]

**Fact:** If \( \lim_{x \to c} g(x) = b \) and \( f(x) \) is (2-sided) continuous at \( b \), then 
\[
\lim_{x \to c} f(g(x)) = f(b).
\]

So if \( g(x) \) is continuous at \( c \) and \( f(x) \) is continuous at \( g(b) \), then \((f \circ g)(x)\) is 
continuous at \( c \).

Where is \( \sin \left( \sqrt{x^2 + 1} \right) \) continuous?

If \( f \) has a removable discontinuity at \( c \) with \( f(c) \) not defined and limit \( L \), then 
\[
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq c, \\ L & \text{if } x = c. \end{cases}
\]
is the *continuous extension* of \( f(x) \) to \( c \).
Define \( g(4) \) in a way that extends
\[
g(x) = \frac{x^2 - 16}{x^2 - 3x - 4}
\]
to be continuous at \( x = 4 \).

For \( x \neq 4 \)
\[
g(x) = \frac{x^2 - 16}{x^2 - 3x - 4} = \frac{(x + 4)(x - 4)}{(x + 1)(x - 4)}
\]
\[
= \frac{x + 4}{x + 1}.
\]
So
\[
\lim_{x \to 4} g(x) = \lim_{x \to 4} \frac{x + 4}{x + 1} = \frac{8}{5}.
\]
If \( g(4) = \frac{8}{5} \), \( g(x) \) is continuous at 4.

For what value of \( b \) is
\[
g(x) = \begin{cases} 
  x & \text{if } x < -2 \\
  bx^2 & \text{if } x \geq -2.
\end{cases}
\]
continuous at every \( x \)?

If \( c \neq -2 \), \( g(x) \) is continuous at \( c \). Since \( g(x) \) is right continuous at \( -2 \), we must find \( b \) so that \( g(x) \) is left continuous at \( -2 \), that is
\[
\lim_{x \to -2^-} g(x) = g(-2) = 4b.
\]
Since
\[
\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x = -2,
\]
we need \( 4b = -2 \), i.e., \( b = -\frac{1}{2} \).
Intermediate Value Theorem

Theorem

Suppose that \( f(x) \) is continuous on \([a, b]\) and that \( f(a) \neq f(b) \). Then for every number \( N \) between \( f(a) \) and \( f(b) \), there is a point \( c \) between \( a \) and \( b \) such that \( f(c) = N \).

Example

Show that the equation \( x^3 - x^2 + 2x - 3 = 0 \) has a root in the interval \((0, 2)\), that is, there is number \( c \) in \((0, 2)\) such that \( c^3 - c^2 + 2c - 3 = 0 \).

We apply the IVT to the function \( f(x) = x^3 - x^2 + 2x - 3 \) on the interval \([0, 2]\). Since \( f(x) \) is a polynomial it is continuous everywhere and so on the interval. We have \( f(0) = -3 \) and \( f(2) = 5 \). So \( N = 0 \) lies between \( f(0) \) and \( f(2) \) (since \(-3 < 0 < 5\)). We may conclude that there is a \( c \) between 0 and 2 such that \( f(c) = 0 \).