2.6: Limits at Infinity: Horizontal Asymptotes

If \( f(x) \) goes to \( L \) as \( x \) goes to \( \infty \), write
\[
\lim_{x \to \infty} f(x) = L.
\]
\[
\lim_{x \to -\infty} f(x) = L
\]
defined similarly.

Clearly, we have
\[
\lim_{x \to \infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x} = 0.
\]

The usual limit laws apply so, for example
\[
\lim_{x \to \infty} \frac{2}{x} - \frac{5}{x^2} = 3 + 2 \cdot 0 + 5 \cdot 0 = 3.
\]
Similarly
\[
\lim_{x \to -\infty} \frac{2}{x} - \frac{5}{x^2} = 3.
\]

Find the limit at \( \infty \) for
\[
f(x) = \frac{7x + 2}{3x - 5}.
\]

Use a simple algebraic trick
\[
f(x) = \frac{(7x + 2)\frac{1}{x}}{(3x - 5)\frac{1}{x}} = \frac{7 + 2/x}{3 - 5/x},
\]
\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{7 + 2/x}{3 - 5/x} = \frac{7}{3}.
\]

There are also infinite limits at \( \pm \infty \).
Observe that
\[
\lim_{x \to -\infty} e^x = 0 = \lim_{x \to \infty} e^{-x}
\]
and
\[
\lim_{x \to \infty} e^x = \infty = \lim_{x \to -\infty} e^{-x}.
\]

Examples
Find the limits of the given funs. at \(\pm\infty\):

- \(f(x) = \frac{2x^2 - 3x + 4}{3x^2 + 7x + 11}\)
- \(g(x) = \frac{13x + 5}{x^2 + 2x + 4}\)

(Ans. \(\frac{2}{3}, 0\))

Use the Sandwich Theorem to show
\[
\lim_{x \to \infty} \frac{2\cos x}{x} = 0.
\]

The line \(y = L\) is a horizontal asymptote of \(y = f(x)\) if either
\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.
\]

The line \(y = 1\) is a horizontal asymptote for the graph \(y = f(x)\).
\[
\lim_{x \to 0} \frac{1}{x} \text{ does not exist but } \\
\lim_{x \to 0^+} \frac{1}{x} = \infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.
\]

Infinite 1-sided limits: write

a. \( \lim_{x \to c^+} f(x) = \infty \) if \( f(x) \) goes to infinity as \( x \) goes to \( c \) from the right,

b. \( \lim_{x \to c^-} f(x) = \infty \) if \( f(x) \) goes to infinity as \( x \) goes to \( c \) from the left.

If both hold, write \( \lim_{x \to c} f(x) = \infty \).

Similar definitions hold for \( -\infty \).

In all the above cases we say \( x = c \) is a vertical asymptote of \( y = f(x) \).

Note that

\[
\lim_{x \to 1^+} \frac{x}{x - 1} = \infty \\
\lim_{x \to 1^-} \frac{x}{x - 1} = -\infty \\
\lim_{x \to \pm \infty} \frac{x}{x - 1} = 1.
\]

We have a vertical asymptote \( x = 1 \) and a horizontal asymptote \( y = 1 \).
Often get an infinite limit if the bottom goes to 0 and the top goes to a nonzero limit. To determine the sign look at the sign of the expression.

In the last example the sign changed from negative to positive. Consider the example:

\[ f(x) = \frac{x}{(x + 1)^2} \]

The numerator goes to \(-1\) and the denominator goes to zero as \(x\) goes to \(-1\), but the expression is negative when \(x\) is near \(-1\). So both 1-sided limits are \(-\infty\).

And we have

\[ \lim_{x \to -1} \frac{x}{(x + 1)^2} = -\infty. \]

Now let \( g(x) = \frac{x}{x^2 - 1} \).

\[ \lim_{x \to 1^+} \frac{x}{x^2 - 1} = \lim_{x \to -1^+} \frac{x}{x^2 - 1} = \infty, \]

\[ \lim_{x \to 1^-} \frac{x}{x^2 - 1} = \lim_{x \to -1^-} \frac{x}{x^2 - 1} = -\infty. \]

Note \( g(x) \) changes sign from negative to positive at \(-1\) and \(1\).

Denominator goes to 0 at \(\pm 1\) (but not the numerator). Vertical asymptotes: \(x = \pm 1\).
Facts:

\[ \lim_{x \to 0^+} \ln x = -\infty \quad \lim_{x \to \infty} \ln x = \infty. \]

This is the mirror image of

\[ \lim_{x \to -\infty} e^x = 0. \]

Note that \( \ln (e^{-10}) = -10 \) and \( \ln (e^{-1000}) = -1000. \)

Note \( \tan x = \frac{\sin x}{\cos x}, \quad \lim_{x \to \pi/2} \cos x = 0 \) and \( \tan x \) changes sign at \( \pi/2 \) from positive to negative. So

\[ \lim_{x \to \pi/2^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to \pi/2^+} \tan x = -\infty. \]
If there is time, find the limit

$$\lim_{x \to -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}.$$

First find the limit of the expression $g(x)$ within the parentheses.

$$g(x) = \frac{x^2 + x - 1}{8x^2 - 3} = \frac{(x^2 + x - 1)\frac{1}{x^2}}{(8x^2 - 3)\frac{1}{x^2}} = \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}}.$$

We have

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}} = \frac{1 + 0 - 0}{8 - 0} = \frac{1}{8}.$$

Hence, by the root law applied to limits at $-\infty$.

$$\lim_{x \to -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3} = \lim_{x \to -\infty} (g(x))^{1/3} = \left( \frac{1}{8} \right)^{1/3} = \frac{1}{2}.$$

Show that

$$\lim_{x \to \pm\infty} \frac{x^3 - 4x}{x^2 - 3x + 2} = \pm\infty.$$
Find the limits of \( \frac{2x}{\sqrt{x^2+3}} \) at \( \pm \infty \).

If \( x > 0 \) we have

\[
\frac{2x}{\sqrt{x^2+3}} = \frac{2x}{\sqrt{x^2+3}} \frac{1}{x} = \frac{2}{\sqrt{1 + \frac{3}{x^2}}}
\]

So \( \lim_{x \to \infty} \frac{2x}{\sqrt{x^2+3}} = 2 \). But if \( x < 0 \) we have

\[
\frac{2x}{\sqrt{x^2+3}} = \frac{2x}{\sqrt{x^2+3}} \frac{1}{x} = \frac{-2}{\sqrt{1 + \frac{3}{x^2}}}
\]

So \( \lim_{x \to -\infty} \frac{2x}{\sqrt{x^2+3}} = -2 \).

Show that

\[
\lim_{x \to \infty} \sqrt{x^2 + 5x - x} = \frac{5}{2}.
\]

Hint: Rationalize. That is, multiply the top and the bottom by \( \sqrt{x^2 + 5x + x} \).

What about the following limit?

\[
\lim_{x \to -\infty} \sqrt{x^2 + 5x - x}
\]

What are the limits of \( \arctan x \) at \( \pm \infty \)?

\[
y = \frac{\pi}{2}, \quad y = -\frac{\pi}{2}
\]