4.2: The Mean Value Theorem

**Theorem (MVT):**
Suppose that \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there is a point \( c \) in \((a, b)\) s.t.

\[
\frac{f(b) - f(a)}{b - a} = f'(c) \quad (\star)
\]

The secant connecting \((a, f(a))\) to \((b, f(b))\) is parallel to the tangent line at \((c, f(c))\).

**Ex a:** \( f(x) = x^2 - 2x + 3, \ [a, b] = [1, 4] \).
Note \( f(x) \) is continuous on \([a, b]\) and diff. on \((a, b)\). Find \( c \) so that \((\star)\) holds.

\[
\frac{f(4) - f(1)}{4 - 1} = \frac{3}{3} = 1;
\]

Solve \( f'(c) = 2c - 2 = 3 \) (Ans \( c = \frac{5}{2} \)).

**Rolle’s Theorem:** special case of the MVT with \( f(a) = f(b) \); in this case, the secant has slope 0.

**MVT follows from Rolle’s Theorem.**
**Ex b:** Show that $f(x) = \sin(2x)$ satisfies the hypotheses of Rolle’s Theorem on $[\frac{\pi}{8}, \frac{3\pi}{8}]$ and find $c$ in $(\frac{\pi}{8}, \frac{3\pi}{8})$ s.t. $f'(c) = 0$. Observe that $f(x)$ is diff. on $(-\infty, \infty)$ and so it is continuous on $[\frac{\pi}{8}, \frac{3\pi}{8}]$ and diff. on $(\frac{\pi}{8}, \frac{3\pi}{8})$. Moreover, $f\left(\frac{\pi}{8}\right) = f\left(\frac{3\pi}{8}\right) = \frac{1}{\sqrt{2}}$. So the hypotheses are satisfied. We have $f'(c) = 2\cos(2c) = 0$ for $c = \frac{\pi}{4}$ in $(\frac{\pi}{8}, \frac{3\pi}{8})$.

**Ex c:** Show that for $g(x) = x^{2/3}$ on $[-8, 8]$, we have $g(-8) = g(8)$ but there is no $c$ in $(-8, 8)$ s.t. $g'(c) = 0$. Why doesn’t this violate Rolle’s Theorem?

$g(8) = 8^{2/3} = 4 = (-8)^{2/3} = g(-8)$ and $g'(x) = \frac{2}{3}x^{-1/3}$. So $g'(x) \neq 0$ for all $x \neq 0$ but $g'(0)$ is not defined! Thus $g(x)$ is not diff. on $(-8, 8)$ and so the hypotheses of Rolle’s Theorem aren’t satisfied.

**Ex d:** Show that $f(x) = x^3 + 3x + 1$ satisfies the hypotheses of the MVT on $[-1, 2]$ and find $c$ in $(-1, 2)$ so that $(\star)$ holds. Note that $f(x)$ is a polynomial and thus continuous and differentiable on $(-\infty, \infty)$. The hypotheses of the MVT are satisfied (with $a = -1$ and $b = 2$). We have $\frac{f(2) - f(-1)}{2 - (-1)} = \frac{15 - (-3)}{3} = 6$.

We have $f'(x) = 3x^2 + 3 = 6$ if $x = 1$. If we take $c = 1$ in $(-1, 2)$, $(\star)$ holds.

**Ex e:** Show that $f(x) = x^3 + 3x + 1 = 0$ has at most one root. If $f(a) = f(b)$ for $a < b$, then by Rolle’s Theorem, $f'(c) = 0$ for some $c$ in $(a, b)$. But since $f'(x) = 3x^2 + 3 > 0$ for all $x$, this is impossible.
Fact: Suppose that $f(x)$ is continuous on $[a, b]$ and suppose that $f'(x) \leq M$ for all $x$ in $(a, b)$. Then $f(b) - f(a) \leq M(b - a)$.

Ex f: Suppose that $f(x)$ is differentiable and $f'(x) \leq 2$ for all real $x$. Given that $f(3) = 5$ show that $f(7) \leq 13$ and $f(0) \geq -1$.

By the above fact we have

$$f(7) - f(3) \leq 2(7 - 3) = 8;$$

so $f(7) \leq f(3) + 8 = 13$. Similarly,

$$f(3) - f(0) \leq 2(3 - 0) = 6;$$

thus $f(0) \geq f(3) - 6 = -1$.

Ex g: Show that for all $a < b$

$$|\cos b - \cos a| \leq b - a.$$ 

Apply the MVT to $g(x) = \cos x$ on $[a, b]$.

Physical interpretation of MVT

If position $s(t)$ is differentiable, then the average velocity on $[a, b]$ is equal to the instantaneous velocity at some point $c$ in $(a, b)$.

The MVT has consequences

i. If $f'(x) > 0$ on $(a, b)$, then $f(x)$ is strictly increasing on $(a, b)$.

ii. If $f'(x) < 0$ on $(a, b)$, then $f(x)$ is strictly decreasing on $(a, b)$.

iii. If $f'(x) = 0$ on $(a, b)$, then there is a constant $c$ so that $f(x) = c$ for all $x$ in $(a, b)$.

Fact: If $f'(x) = g'(x)$ on the interval $(a, b)$ then $f(x) = g(x) + c$ on $(a, b)$ for some constant $c$. 