**4.7: Optimization Problems**

**Problem**
A farmer wants to enclose a rectangular plot with 60 m of fencing. Find the dimensions of the plot that maximize the area enclosed. Find the resulting area.

Let $x$, $y$ denote the dimensions of the rectangular plot. We need to maximize the area $A = xy$ subject to the constraint $2x + 2y = 60$.

Get a function of 1-variable $A(x)$, by first using (**) to solve for $y$

\[ y = \frac{1}{2} (60 - 2x) = 30 - x, \]

then substitute into (*)

\[ A(x) = xy = x(30 - x) = 30x - x^2. \]

Since both dimensions are positive we get $x \geq 0$ and $30 - x = y \geq 0$. So $0 \leq x \leq 30$.

Now use the Extreme Value Theorem.

\[ A'(x) = 30 - 2x = 0 \quad \text{if } x = 15 \]

We have

\[ A(0) = A(30) = 0 \quad \text{abs min} \]
\[ A(15) = 225 \quad \text{abs max} \]

Thus the dimensions of the plot are $x = 15$ m and $y = 30 - x = 15$ m.

Maximum area: $A(15) = 225$ m$^2$
Methodology for optimization

i. Read the problem carefully; make a sketch and assign variable names.

ii. What is to be optimized (max or min)?

iii. Write down relevant equations. Use constraint(s) to eliminate variable(s). Get function of 1 variable.

iv. Use calculus to find max or min on the relevant interval.

Ex a: Find the point on the line $2x + y = 2$ that lies closest to $(0, 0)$.

The distance from $(0, 0)$ to $P(x, y)$ is $d = \sqrt{x^2 + y^2}$; if $P$ lies on the line then $2x + y = 2$ (constraint). Use the constraint to solve for $y$: $y = 2 - 2x$.

Then:

Then:

\[
d(x) = \sqrt{x^2 + (2 - 2x)^2} = \sqrt{5x^2 - 8x + 4}
\]

\[
d'(x) = \frac{5x - 4}{\sqrt{5x^2 - 8x + 4}} = 0
\]

if $x = 4/5$.

Sign of $f'(x)$:

\[
\begin{array}{c|c}
\hline
& - & + \\
\hline
4/5 & & \\
\hline
\end{array}
\]

Thus $d(x)$ has an abs. minimum at $x = 4/5$. So the point on the line $2x + y = 2$ that lies closest to the origin is $(4/5, 2 - 2(4/5)) = (4/5, 2/5)$.

Distance to $(0, 0)$: $d(4/5) = 2/\sqrt{5}$.
Ex b: A square-bottom box with no top is to have a volume of 500 cc. What dimensions minimize surface area? Find the resulting surface area.

Let $x$ denote the edge length of the square bottom and let $y$ denote the height of the box. We have

$$V = x^2y = 500.$$ 

Surface Area

$$S = x^2 + 4xy.$$ 

Constraint:

$$x^2y = 500$$

Solve: $y = \frac{500}{x^2}$

Then for $x > 0$ we have

$$S(x) = x^2 + \frac{2000}{x}.$$ 

Now differentiate:

$$S'(x) = 2x - \frac{2000}{x^2}$$

$$= \frac{2(x^3 - 1000)}{x^2} = 0$$

if $x = 10$.

Note $S'(5) = -70$, $S'(20) = 35$

Sign of $S'(x)$: $-\frac{10}{+}$

Abs. min: $S(10) = 300$

The dimensions which minimize surface area:

$x = 10$ cm and

$y = \frac{500}{x^2} = \frac{500}{100} = 5$ cm.

Resulting surface area 300 cm$^2$
Ex c: Find the dimensions of the cylinder with maximum volume that can be inscribed in a hemisphere of radius 3. Find the volume.

The relevant equations are:

\[ V = \pi r^2 h \]
maximize

\[ r^2 + h^2 = 9 \]
constraint

We have \( r = \sqrt{9 - h^2} \) for \( 0 \leq h \leq 3 \).

\[
V(h) = \pi r^2 h = \pi (9 - h^2) h = \pi (9h - h^3)
\]

\[
V'(h) = 3\pi (3 - h^2) = 0 \text{ if } h = \pm \sqrt{3}.
\]

Note that \( h \geq 0 \).

Sign of \( V'(h) \):

\[
\begin{array}{c|c|c}
0 & \sqrt{3} & -\\
\hline
+ & -
\end{array}
\]

So \( V(\sqrt{3}) = 6\sqrt{3}\pi \) max volume, since \( V(0) = V(3) = 0 \).

Dimensions of the cylinder with maximum volume: \( h = \sqrt{3}, \ r = \sqrt{9 - h^2} = \sqrt{6} \).

Ex d: Find the dimensions of the rectangle in the 1\(^{st}\) quadrant with one corner at the origin and the opposite corner on the line \( 2x + 3y = 12 \) which maximize area. Find the area.

\[
V'(h) = 3\pi (3 - h^2) = 0 \text{ if } h = \pm \sqrt{3}.
\]

So \( V(\sqrt{3}) = 6\sqrt{3}\pi \) max volume, since \( V(0) = V(3) = 0 \).

Dimensions of the cylinder with maximum volume: \( h = \sqrt{3}, \ r = \sqrt{9 - h^2} = \sqrt{6} \).
Ex e: Design a poster with 50 cm$^2$ of printing, 2 cm side margins and 4 cm top/bottom margins using the least paper. What are its dimensions?

Let $x, y$ denote the width and height of the printed area.

Here is a sketch illustrating the problem:

Eliminate $y$ using $y = 50/x$.

$$A(x) = x\frac{50}{x} + 8x + 4\frac{50}{x} + 32$$
$$= 8x + \frac{200}{x} + 82, \quad x > 0$$

$$A'(x) = 8 - \frac{200}{x^2} = \frac{8(x^2 - 25)}{x^2} = 0$$

if $x = \pm 5$; but only $x = 5 > 0$ is relevant to the problem.

Sign of $A'(x)$: $\overbrace{-}^{5} +$

So the absolute min. is $A(5) = 162$ cm$^2$ and the dimensions of the poster are $9 \text{ cm} \times 18 \text{ cm}$.
Ex f: Find the dimensions of an open top can with volume $512 \pi \text{ cm}^3$ which minimizes surface area.

Volume of a cylinder is $V = \pi r^2 h$.

![Diagram of a cylinder with dimensions labeled]

The surface area of the open top can is the sum of the side area and the area of the base.

$$S = \pi r^2 + 2\pi rh$$

Need to minimize the surface area subject to the constraint that

$$V = \pi r^2 h = 512\pi.$$

Use this to solve for $h$.

We have $h = 512/r^2$. Then we can write $S$ as a function of $r$:

$$S(r) = \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r \frac{512}{r^2}$$

$$= \pi r^2 + \frac{1024\pi}{r} \quad r > 0$$

Now differentiate and set equal to 0

$$S'(r) = 2\pi r - \frac{1024\pi}{r^2}$$

$$= \frac{2\pi(r^3 - 512)}{r^2} = 0$$

if $r = 8$. Sign of $S'(r)$: 

$\frac{1}{8}$

The dimensions of the cylinder with min surface area: $r = h = 8 \text{ cm}$ and its. surface area is $S(8) = 192\pi \text{ cm}^2$. 
Ex g: An open box is to be constructed from a 45 cm $\times$ 24 cm rectangular piece of sheet metal by cutting squares off each corner and folding up the sides. Find the volume of the largest such box and its dimensions.

Let $x$, $y$, $z$ denote the dimensions and $V$ the volume of the box. Then 

$V = xyz$, $x + 2z = 45$, $y + 2z = 24$

We want to maximize the volume. Eliminate $x, y$ using the constraints. For $0 \leq z \leq 12$

$V(z) = (45 - 2z)(24 - 2z)z$ 

$= 4z^3 - 138z^2 + 1080z$

$V'(z) = 12z^2 - 276z + 1080 = 0$.

Use the quadratic formula

$z = \frac{276 \pm \sqrt{(-276)^2 - 4 \cdot 12 \cdot 1080}}{2 \cdot 12} = 5, 18$.

Only $z = 5$ is in the interval.

$V(0) = V(12) = 0$

$V(5) = 2450$ abs max

Thus the largest such box has volume 2450 cm$^3$ and it has dimensions 35 cm $\times$ 14 cm $\times$ 5 cm.
Ex h: Find the volume of the largest cone that can be inscribed in a sphere of radius 3 and find its dimensions.

Volume of a cone is $V = \frac{1}{3} \pi r^2 h$.

Note that $r^2 + x^2 = 9$ and $h = x + 3$.

Should we eliminate $x$ or $r$?

Either $x = \sqrt{9 - r^2}$ or $r = \sqrt{9 - x^2}$:

$$V(r) = \frac{\pi r^2}{3} (\sqrt{9 - r^2} + 3), \quad [0, 3]$$

$$V(x) = \frac{\pi}{3} (9 - x^2)(3 + x), \quad [0, 3]$$

Let’s use the second expression (it looks simpler). Multiplying out and differentiating we get:

$$V(x) = -\frac{\pi}{3} (x^3 + 3x^2 - 9x - 27)$$

$$V'(x) = -\pi (x^2 + 2x - 3) = -\pi (x - 1)(x + 3) = 0$$

if $x = 1, -3$. Only $x = 1$ is in $(0, 3)$.

Use the Extreme Value Theorem: (could also use a sign analysis)

$V(0) = 9\pi, \ V(3) = 0$ and $V(1) = 32\pi/3$ abs max.

So the volume of the largest cone is $32\pi/3$.

The corresponding dimensions are:

$r = \sqrt{9 - x^2} = \sqrt{8}$ and $h = x + 3 = 4$. 