4.9: Antiderivatives

Fact: If $f'(x) = g'(x)$ on the interval $(a, b)$ then $f(x) = g(x) + c$ on $(a, b)$ for some constant $c$.

Terminology: If $F'(x) = f(x)$, then $F(x)$ is called an antiderivative of $f(x)$. The most general antiderivative is $F(x) + C$, $C$ arbitrary constant

We use antiderivatives to evaluate integrals.

Physics: Given velocity $v$ and initial position $s_0$, we would like to find the position $s$.

Examples: Find the general antiderivative:

a: Let $f(x) = x$; then $F(x) = x^2/2 + C$.

b: Let $f(x) = 3 \sin 2x$; then $F(x) = -\frac{3}{2} \cos 2x + C$.

Basic fact: Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$ and let $a, b$ be constants. The general antiderivative of $af(x) + bg(x)$ is given by

$$aF(x) + bG(x) + C$$

So we can find antiderivatives termwise just like derivatives.

More examples:

Find the general antiderivative:

c: Let $f(x) = 4x - 3 - 2e^{-5x}$, then $F(x) = 2x^2 - 3x + \frac{2}{5} e^{-5x} + C$.

d: $f(x) = 3 \cos \pi x + 5e^{3x} - \frac{2}{x^4}$

$F(x) = \frac{3}{\pi} \sin \pi x + \frac{5}{3} e^{3x} + \frac{2}{3x^3} + C$
Some basic antiderivatives (take $k = 1$ or any nonzero constant):

\[
\begin{align*}
f(x) &= x^n, \quad n \neq -1 \\
f(x) &= \frac{1}{x} \\
f(x) &= e^{kx} \\
f(x) &= \cos kx \\
f(x) &= \sin kx \\
f(x) &= \sec^2 kx \\
f(x) &= \frac{1}{k^2 + x^2} \\
f(x) &= \frac{1}{\sqrt{k^2 - x^2}} \\
f(x) &= a^{kx}, \quad a > 0, \; a \neq 1
\end{align*}
\]

\[
\begin{align*}
F(x) &= \frac{x^{n+1}}{n+1} + C \\
F(x) &= \ln |x| + C \\
F(x) &= \frac{1}{k} e^{kx} + C \\
F(x) &= \frac{1}{k} \sin kx + C \\
F(x) &= -\frac{1}{k} \cos kx + C \\
F(x) &= \frac{1}{k} \tan kx + C \\
F(x) &= \frac{1}{k} \tan^{-1} \frac{x}{k} + C \\
F(x) &= \sin^{-1} \frac{x}{k} + C \\
F(x) &= \frac{1}{k \ln a} a^{kx} + C
\end{align*}
\]
Given a function $f(x)$ we are often interested in finding the antiderivative $F(x)$ s.t. and $F(x_0) = y_0$ for some $x_0$; this is the specific antiderivative of $f(x)$ whose graph passes through the point $(x_0, y_0)$.

**Ex e:** Given $f(x) = 4x + 3e^{-x}$, find the antiderivative $F(x)$ such that $F(0) = 2$.
The general antiderivative is $F(x) = 2x^2 - 3e^{-x} + C$; we want $F(0) = 2$. Thus $2 = F(0) = 0 - 3e^{-0} + C = C - 3$ and so $C = 5$. Hence, $F(x) = 2x^2 - 3e^{-x} + 5$.

In physics, we may be given the velocity $v(t)$ of an object as a function of time $t$ and asked to find its position $s(t)$ if its position at $t_0$ is $s(t_0) = s_0$. Since $v(t) = s'(t)$, this is really the same kind of problem as the one we just did.

**Ex f:** Find $s(t)$, given $v(t) = 2\sin 3t$ and $s(\pi/3) = 1$.

$$s(t) = -\frac{2}{3} \cos 3t + C$$

Thus $s(t) = -\frac{2}{3} \cos 3t + \frac{1}{3}$.

If we know the acceleration $a(t) = s''(t)$ instead of the velocity $v(t)$, we need to find the antiderivative twice to get $s(t)$!

**Ex g:** Suppose $a(t) = 12t$, $v(1) = 3$ and $s(1) = 6$. Find $s(t)$.

$v(t) = 6t^2 + C = 6t^2 - 3$ since $v(1) = 3$.

Since $s(t)$ is an antiderivative of $v(t)$ we have $s(t) = 2t^3 - 3t + C'$; since $s(1) = 6$ we have $C' = 7$ so

$$s(t) = 2t^3 - 3t + 7.$$
**Ex h:** Given an object’s velocity is 
\[ v(t) = (10t + 7) \text{ m/s} \] and its initial position is \( s(1) = 3 \text{ m} \), find \( s(t) \).

Recall that \( v(t) = s'(t) \). Since
\[ \frac{d}{dt}(5t^2 + 7t) = 10t + 7, \]
we have \( s(t) = 5t^2 + 7t + c \).

We need \( s(1) = 3 \); substituting \( s(1) = 5 + 7 + c = 12 + c \). Solve \( 3 = 12 + c \) to get \( c = -9 \).
Thus \( s(t) = (5t^2 + 7t - 9) \text{ m} \).

Here are a few more examples.

**Ex i:** Find \( f(x) \) so that \( f'(x) = \frac{6}{2x-3} \) and \( f(2) = 5 \).

Ans: \( f(x) = 3 \ln(2x - 3) + 5 \).

**Ex j:** Find \( f(x) \) so that \( f'(x) = 2\sqrt{x} \) and its graph passes through the point \((1, 2)\).

Ans: \( f(x) = \frac{4x^{3/2}+2}{3} \).

**Ex k:** Suppose we are given that
acceleration \( a(t) = 12 \cos 4t \text{ m/s}^2 \), initial velocity \( v(0) = 6 \text{ m/s} \) and initial position \( s(0) = 1 \text{ m} \). Find position.

Ans: \( s(t) = (-\frac{3}{4} \cos 4t + 6t + \frac{7}{4}) \text{ m} \).

**Ex l:** Find a function \( f \) so that \( f'''(t) = 6 \) and \( f(1) = f'(1) = f''(1) = 0 \).

Ans: \( f(x) = x^3 - 3x^2 + 3x - 1 \).