5.2: The Definite Integral

Let $f(x)$ be a function defined and bounded on a closed interval $[a, b]$. The definite integral of $f(x)$ over $[a, b]$ is defined to be the limit of the Riemann sums we used to approximate the area under the graph $y = f(x)$ from $a$ to $b$ (where $f(x) \geq 0$ on $[a, b]$) provided that the limit exists. The only real difference is that we no longer require $f(x) \geq 0$ on $[a, b]$.

We subdivide the interval $[a, b]$ into $n$ subintervals of length $\Delta x = (b - a)/n$: $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $x_0 = a$ and $x_n = b$. Let $c_i = \frac{1}{2}(x_{i-1} + x_i)$ be the midpoint of the $i^{th}$ subinterval $[x_{i-1}, x_i]$.

We define the left, right and mid-point sums as before:

$$L_n = (f(x_0) + \cdots + f(x_{n-1})) \Delta x$$
$$= \sum_{i=1}^{n} f(x_{i-1}) \Delta x$$

$$R_n = (f(x_1) + \cdots + f(x_n)) \Delta x$$
$$= \sum_{i=1}^{n} f(x_i) \Delta x$$

$$M_n = (f(c_1) + \cdots + f(c_n)) \Delta x$$
$$= \sum_{i=1}^{n} f(c_i) \Delta x$$
Try exercises 5.2.3, 5.2.8, 5.2.33. Note that if \( f(x) \) is increasing \( L_n \) is the lower sum and \( R_n \) is the upper sum (v. v. if \( f(x) \) decr.). Suppose \( f(x) \) is continuous on \([a, b]\). Then the limits of \( L_n, R_n \) and \( M_n \) all exist and are the same! We write:

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} M_n
\]

Limit is called the \textit{definite integral} of \( f(x) \) over \([a, b]\). \( f(x) \) is called the integrand, \( x \) variable of integration, and \( a, b \) are the limits of integration.

Say that \( f(x) \) is \textit{integrable} on \([a, b]\) if upper sums and lower sums converge to the same limit. So if \( f(x) \) is continuous on \([a, b]\), then \( f(x) \) is integrable on \([a, b]\).

**Fact** If \( f(x) \geq 0 \) on \([a, b]\), then the area under the curve is given by

\[
A = \int_a^b f(x) \, dx.
\]

**Ex a:** Find the area under the curve \( y = x \) over \([1, 3]\). Use the fact that the area of a triangle is \( \frac{1}{2}bh \):

\[
A = \int_1^3 x \, dx = \frac{1}{2} 3^2 - \frac{1}{2} 1^2 = 4
\]
More generally,
\[
\int_a^b f(x) \, dx = A_1 - A_2,
\]

\(A_1\) is the area below the graph and above the \(x\)-axis and \(A_2\) is the area above the graph and below the \(x\)-axis.

Ex b: Evaluate \(\int_1^4 2x - 4 \, dx\)

Note \(f(x)\) changes sign at \(x = 2\).
\[
\int_1^4 2x - 4 \, dx = \frac{2 \cdot 4}{2} - \frac{1 \cdot 2}{2} = 3
\]

Ex c: Evaluate \(\int_{-5}^5 \sqrt{25 - x^2} \, dx\)

Area of the region: \(25\pi/2\). So
\[
\int_{-5}^5 \sqrt{25 - x^2} \, dx = A = \frac{25\pi}{2}
\]

Ex d:
\[
\int_{-1}^4 \frac{1 - x}{2} \, dx = \frac{1}{2} \cdot 2 \cdot 1 - \frac{1}{2} \cdot 3 \cdot \frac{3}{2} = -\frac{5}{4}
\]
Basic rules of definite integrals

i. If $a < b$, define $\int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx$.

ii. Define $\int_{a}^{a} f(x) \, dx = 0$.

iii. For a constant $k$, $\int_{a}^{b} kf(x) \, dx = k \int_{a}^{b} f(x) \, dx$.

iv. $\int_{a}^{b} f(x) + g(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$.

v. $\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx$ (see Ex b, d above).

vi. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx$.

(Linearity) Combining rules (iii) and (iv) we get

$$\int_{a}^{b} cf(x) + dg(x) \, dx = c \int_{a}^{b} f(x) \, dx + d \int_{a}^{b} g(x) \, dx$$
Some special formulas:
\[ \int_a^b c \, dx = c(b - a), \quad \int_a^b x \, dx = \frac{b^2 - a^2}{2}, \quad \int_a^b x^2 \, dx = \frac{b^3 - a^3}{3} \]

More examples:
\[ \int_2^5 (4x - 7) \, dx = 4 \int_2^5 x \, dx - \int_2^5 7 \, dx = 4 \frac{5^2 - 2^2}{2} - 7(5 - 2) = 21 \]
\[ \int_0^6 |x - 2| \, dx = \int_0^2 (2 - x) \, dx + \int_2^6 (x - 2) \, dx \]
\[ = 2(2 - 0) - \frac{2^2 - 0^2}{2} + \frac{6^2 - 2^2}{2} - 2(6 - 2) = 10 \]
\[ \int_1^4 5x^2 + 2 \, dx = 5 \int_1^4 x^2 \, dx + \int_1^4 2 \, dx = 5 \frac{4^3 - 1^3}{3} + 2(4 - 1) = 111 \]

The average value of \( f(x) \) on \([a, b]\) is \( \mu = \frac{1}{b - a} \int_a^b f(x) \, dx \).

Find avg. value of \( f(x) = 5x^2 + 2 \) on \([1, 4]\): \( \mu = \frac{1}{4 - 1} \int_1^4 5x^2 + 2 \, dx = 37 \).