1.2 - The Natural Numbers

Let denote the set of natural numbers, \( \mathbb{N} = \{1, 2, 3, \ldots \} \). \( \mathbb{N} \) can be axiomatized using the Peano postulates (or axioms).

- \( N1: 1 \in \mathbb{N} \)
- \( N2: \) If \( n \in \mathbb{N} \), then \( n+1 \in \mathbb{N} \) (every \( n \) has a successor).
- \( N3: \forall n \in \mathbb{N}, \, 1 \neq n+1 \) (1 is not a successor).
- \( N4: \forall m, n \in \mathbb{N}, \, m+1 = n+1 \Rightarrow m = n \) (so \( n \mapsto n+1 \) injective).
- \( N5: \) Let \( A \subseteq \mathbb{N} \) and \( \text{Sp} \) such that
  \[ (i) \, \, \, 1 \in A \]
  \[ (ii) \, \, \, \forall n \in \mathbb{N}, \, n \in A \Rightarrow n+1 \in A \]
  Then \( A = \mathbb{N} \).

Theorem (Principle of Mathematical Induction).

Let \( P_1, P_2, P_3, \ldots \) be a sequence of statements. \( \text{Sp} \) such that

- \( i) \, \, \, P_1 \) true (base step),
- \( ii) \, \, \forall n, \, P_n \Rightarrow P_{n+1} \) (inductive step).

Then \( P_n \) is true for all \( n \in \mathbb{N} \).

Sketch of proof: Apply \( N5 \) to \( A = \{ n \in \mathbb{N} : P_n \} \).

ex.1. Prove \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \) for all \( n \in \mathbb{N} \).

Recursive definition of a sequence.

Proposition: Let \( f: X \to X \) be a function and let \( x \in X \).

Then there is a unique sequence \( \{ x_n \}_{n \in \mathbb{N}} \) in \( X \) (\( x_1, x_2, x_3, \ldots \in X \)) such that \( x_1 = x \) and \( x_{n+1} = f(x_n) \) for all \( n \in \mathbb{N} \).

ex.2. (see 1.2.12) Let \( x \in (0, 2) \) and let \( \{ x_n \}_{n \in \mathbb{N}} \) be defined recursively by \( x_1 = x \) and \( x_{n+1} = \sqrt{x_n + 2} \). Show that \( 0 < x_n < x_{n+1} < 2 \) for all \( n \in \mathbb{N} \).

To solve the problem take \( x = (0, 2) \) and show that \( f(x) = \sqrt{x+2} \in X \) for all \( x \in (0, 2) \). This gives \( f: X \to X \).

Next show that \( x \leq f(x) \) for all \( x \in X \).

ex.3. (see 1.2.14) Let \( \{ x_n \}_{n \in \mathbb{N}} \) be defined recursively by \( x_1 = 1 \), \( x_{n+1} = \frac{1}{1 + x_n} \). Prove that \( x_{n+2} \) lies between \( x_n \) and \( x_{n+1} \).

Sketch: Prove this inductively.

Observe that \( x_1 = 1 \), \( x_2 = \frac{1}{2} \), \( x_3 = \frac{2}{3} \) so \( x_2 < x_3 < x_1 \). Thus \( x_{n+2} \) lies between \( x_1 \) and \( x_{n+1} \).

For the inductive step, we suppose that the statement holds for some \( n \in \mathbb{N} \). There are two cases to consider:

(a) \( x_n < x_{n+2} < x_{n+1} \)
(b) \( x_{n+1} < x_{n+2} < x_n \)

It is then easy to check that \( x_{n+3} \) lies between \( x_{n+1} \) and \( x_{n+2} \).