1. For \( x \in \mathbb{R} \) define \( F(x) := \int_0^x f(t) \, dt \) where the function \( f \) is given by
\[
f(x) := \begin{cases} 
1 & \text{if } x < 0 \\
3x^2 & \text{if } 0 \leq x < 1 \\
2x + 1 & \text{if } x \geq 1 
\end{cases}
\]
Prove that \( F \) is differentiable at 1 and 2 but not at 0. Find \( F'(1) \). You may assume that \( f \) is integrable on any closed bounded interval \([a, b] \).
Note that \( f \) is integrable on any closed bounded interval \( I \) because \( f \) is continuous at every real number \( \neq 0 \) and is bounded on \( I \) (see 5.2.10). The continuity at \( a = 1 \) follows from the facts that \( f(1) = 3 \) and
\[
\lim_{x \to 1^-} f(x) = 3 = \lim_{x \to 1^+} f(x).
\]
Note that \( f \) is also continuous at 2 (because the function \( x \mapsto 2x + 1 \) is continuous). Thus by the Second Fundamental Theorem of Calculus (see 5.3.3), \( F \) is differentiable at 1 and 2; moreover, \( F'(1) = f(1) = 3 \) and \( F'(2) = f(2) = 3 \).
We now show that \( F \) is not differentiable at 0 by showing that \( \lim_{x \to 0} \frac{F(x) - F(0)}{x} \) does not exist.
\[
\lim_{x \to 0^+} \frac{F(x) - F(0)}{x} = \lim_{x \to 0^+} \frac{1}{x} \int_0^x 3t^2 \, dt = \lim_{x \to 0^+} \frac{x^3}{x} = 0,
\]
\[
\lim_{x \to 0^-} \frac{F(x) - F(0)}{x} = \lim_{x \to 0^-} \frac{1}{x} \int_0^x 1 \, dt = \lim_{x \to 0^-} \frac{x}{x} = 1.
\]
Since the left and right limits are not equal, the limit does not exist and so \( F' \) is not differentiable at 0. \( \square \)

2. Show that the given function \( g \) is differentiable on its natural domain and find its derivative.
\[
g(x) := \int_{x^3}^{e^x} \cos t^2 \, dt \quad \text{for } x \in \mathbb{R}.
\]
Define \( f(x) := \cos x^2 \) for all \( x \in \mathbb{R} \) and observe that \( f \) is continuous on \( \mathbb{R} \) (since it is a composition of two continuous functions). Hence, \( f \) is integrable on every closed bounded interval and so by the Second Fundamental Theorem of Calculus (see 5.3.3) the function given by
\[
F(x) := \int_0^x \cos t^2 \, dt
\]
is differentiable on \( \mathbb{R} \). Moreover, \( F'(x) = f(x) = \cos x^2 \) for all \( x \in \mathbb{R} \). Hence, by the First Fundamental Theorem of Calculus (see 5.3.1) we have
\[
g(x) = \int_{x^3}^{e^x} \cos t^2 \, dt = F(e^x) - F(x^3)
\]
for all \( x \in \mathbb{R} \). Thus, by the Chain Rule (applied twice) and the fact that the difference of differentiable functions is itself differentiable, \( g \) is differentiable. Moreover for all \( x \in \mathbb{R} \)
\[
g'(x) = F'(e^x)e^x - F'(x^3)3x^2 = e^x \cos e^x - 3x^2 \cos x^6
\]
by the Chain Rule and the algebra of derivatives (see 4.2.7 and 4.2.6). \( \square \)

3. Prove that
\[
\lim_{x \to \infty} \exp x = \infty \quad \text{and} \quad \lim_{x \to -\infty} \exp x = 0.
\]
Recall that the function \( f(x) := \exp x \) is strictly increasing on \( \mathbb{R} \). Let \( M > 0 \). Then for all \( x > \ln M \), we have \( \exp x > \exp(\ln M) = M \). Hence,
\[
\lim_{x \to \infty} \exp x = \infty
\]
Let \( \varepsilon > 0 \) (we may assume that \( \varepsilon < 1 \) if necessary). Then for all \( x < \ln \varepsilon \), we have \( \exp x < \exp \ln \varepsilon = \varepsilon \). Therefore,
\[
\lim_{x \to -\infty} \exp x = 0.
\]

4. Prove that the function \( h \) given below has a minimum value and find it.
\[
h(x) := \int_1^x (\ln t)^3 \, dt \quad \text{for all } x > 0.
\]
Since \( f(x) := (\ln x)^3 \) is continuous on \( I := (0, \infty) \), the function \( h \) is differentiable on \( I \). Since \( h'(x) = (\ln x)^3 > 0 \) for \( x > 1 \) and \( h'(x) = (\ln x)^3 < 0 \) for \( 0 < x < 1 \), \( h \) is strictly increasing on \([1, \infty)\) and strictly decreasing on \((0, 1)\).
Hence, \( f(1) \leq f(x) \) for all \( x > 0 \) and therefore \( f(1) = 0 = \min_J f \). \( \square \)
5. Let $f$ be defined on $I := (-\pi/2, \pi/2)$ by $f(x) := \tan x$ for all $x \in I$. Given that $f(I) = \mathbb{R}$ and $f'(x) = \sec^2 x$ for all $x \in I$, prove that $f$ has a differentiable inverse defined on $\mathbb{R}$ and

$$f^{-1}(x) = \int_{0}^{x} \frac{1}{t^2 + 1} \, dt \quad \text{for all } x \in \mathbb{R}.$$ 

We apply Theorem 4.2.9 to $f$ but first we check its hypotheses. Observe that $f$ is differentiable and therefore continuous on $I$ and $f'(x) = \sec^2 x > 0$ for all $x \in I$. Hence $f$ is strictly increasing (and thus strictly monotone) on $I$ (by 4.3.5). Moreover $f'(x) \neq 0$ for all $x \in I$. Thus by 4.2.9, $f$ has a differentiable inverse $f^{-1}$ defined on $f(I) = \mathbb{R}$ and for every $t \in \mathbb{R}$ we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(f^{-1}(x))} = \frac{1}{\tan^2(f^{-1}(x)) + 1} = \frac{1}{x^2 + 1}$$

since $\tan(f^{-1}(x)) = f(f^{-1}(x)) = x$.

By the Second Fundamental Theorem of Calculus (see 5.3.3) the function $F$ defined by

$$F(x) := \int_{0}^{x} \frac{1}{t^2 + 1} \, dt \quad \text{for all } x \in \mathbb{R}$$

is differentiable (since the function $x \mapsto \frac{1}{x^2 + 1}$ is continuous on $\mathbb{R}$) and

$$F'(x) = \frac{1}{x^2 + 1} = (f^{-1})'(x) \quad \text{for all } x \in \mathbb{R}.$$ 

Hence, $F$ and $f^{-1}$ differ by a constant, i.e., there is a constant $c$ such that $f^{-1}(x) = F(x) + c$ for all $x \in \mathbb{R}$. But since $f(0) = 0$, we have $f^{-1}(0) = 0 = F(0)$ and so $c = 0$. Hence, $f^{-1}(x) = F(x)$ for all $x \in \mathbb{R}$. The desired result now follows. \qed