1. Let $A \subset \mathbb{R}$ be a nonempty subset and suppose that $\min A$ exists. Prove that $\inf A = \min A$.

Let $m = \min A$ and observe that $m \leq a$ for all $a \in A$. Hence, $m$ is a lower bound for $A$. Let $\ell$ be a lower bound for $A$; then since $m \in A$, we must have $\ell \leq m$. Hence $\min A = m$ is the greatest lower bound of $A$ and so $\inf A = \min A$. (Alternatively, after observing that $m$ is a lower bound for $A$, one may argue as follows: Let $x > m$; then since $m \in A$, $x$ is not a lower bound for $A$. Therefore $m = \text{glb} A = \inf A$.)

2. Show that the given set is bounded. Find the least upper bound and the greatest lower bound for it; prove your results.

$$A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$$

Let $n \in \mathbb{N}$; since $0 < n < n+1$ we have $0 \leq \frac{n}{n+1} \leq 1$. Hence $A$ is bounded. Note that $\frac{n}{n+1} = 1 - \frac{1}{n+1}$, so if $m < n$, $\frac{m}{m+1} < \frac{n}{n+1}$. Hence $\min A = \frac{1}{1+1} = \frac{1}{2}$ and so by problem 1, $\text{glb} A = \frac{1}{2}$.

We claim that $\text{lub} A = 1$. As noted above, 1 is an upper bound of $A$. Let $x < 1$, then for $n > \frac{1}{1-x}$, we have

$$\frac{n}{n+1} = 1 - \frac{1}{n+1} > x.$$

Therefore $\text{lub} A = 1$ as claimed.

3. Let $B = \{ x \in \mathbb{Q} : x^2 < 2x + 3 \}$. Use the Rational Density Theorem to show that $\sup B = 3$.

(Rational Density Theorem: For every $a, b \in \mathbb{R}$ with $a < b$, there is $r \in \mathbb{Q}$ such that $a < r < b$.)

Since $x^2 < 2x + 3$ iff $(x-3)(x+1) < 0$, we have $B = (-1, 3) \cap \mathbb{Q}$. It is straightforward to check that 3 is an upper bound for $B$. Now let $x < 3$, and set $y = \max\{-1, x\}$. By the Rational Density Theorem, there is $r \in \mathbb{Q}$ such that $y < r < 3$. Since $r > y \geq -1$, $r \in B$ and since $r > y \geq x$, we have $r > x$. Therefore, $\sup B = 3$.

4. Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$. Prove that $\sup S \leq \inf T$.

Let $t \in T$. Then since $s \leq t$ for all $s \in S$, $t$ is an upper bound for $S$; hence $S$ is bounded above and $\sup S \leq t$. Since $t$ was chosen arbitrarily, $\sup S$ is a lower bound for $T$; hence, $T$ is bounded below and $\sup S \leq \inf T$. (Proof by contradiction also works.)

5. Let $f, g$ be $\mathbb{R}$-valued functions defined on a set $A$ and suppose that $f$ and $g$ are bounded on $A$. Prove that $\sup_A (f + g) = \sup_A f + \sup_A g$ without using Theorem 1.5.10(c). If possible find an example for which $\sup_A (f + g) < \sup_A f + \sup_A g$.

Since $f$ and $g$ are bounded on $A$, both $\sup_A f$ and $\sup_A g$ are finite. Recall that $(f + g)(a) = f(a) + g(a)$ for $a \in A$.

Now let $a \in A$, then $f(a) \leq \sup_A f$ and $g(a) \leq \sup_A g$ and hence

$$(f + g)(a) = f(a) + g(a) \leq \sup_A f + \sup_A g.$$ 

Therefore $\sup_A f + \sup_A g$ is an upper bound for $\{(f + g)(a) : a \in A\}$ and so $\sup_A (f + g) \leq \sup_A f + \sup_A g$.

Let $A = [0, 1]$ and let $f, g$ be defined by $f(x) = x$ and $g(x) = 1 - x$ for all $x \in A$. Then $(f + g)(a) = 1$ for all $a \in A$. Moreover, $\sup_A (f + g) = \sup_A f = \sup_A g = 1$ and thus $\sup_A (f + g) = 1 < 2 = \sup_A f + \sup_A g$. □