3.2. Properties of Continuous Functions

Recall: If $x_n$ is a convergent seq. in $[a,b]$, then $\lim x_n \in [a,b]$.

**f:** $D \rightarrow \mathbb{R}$ is bdd on $D$ if $\exists M > 0$ s.t. $|f(x)| \leq M$ for all $x \in D$.

**Lemma:** Sps $f$ is cts on $I = [a,b]$. Then $f$ is bdd on $I$.

**Proof:** Sps not. Then $\forall n, \exists x_n \in I$ s.t. $|f(x_n)| > n$. Since $\{x_n\}_n$ is a bdd seq, it has a convergent subseq $\{x_{n_k}\}_k$ by BWT.

Then $x_0 = \lim x_{n_k} \in I$. Since $f$ is cts on $I$, $f(x_{n_k}) \rightarrow f(x_0)$. So $\{f(x_{n_k})\}_k$ is bdd. But $|f(x_{n_k})| \geq n_k > k$ for each $k$.

This results in a contradiction and so $f$ is bdd. $\square$

**Note:** If $f$ is cts on an interval which is either not closed or not bdd then $f$ need not be bdd on the interval.

**Theorem:** Sps that $f$ is cts on $I = [a,b]$, then $\exists x_0, y_0 \in I$ st.

$f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in I$, that is, $f(x_0) = \inf_I f$ and $f(y_0) = \max_I f = \sup_I f$.

**Proof:** By the Lemma $m = \inf_I f$ and $M = \sup_I f$ are both finite.

We prove $\exists x_0 \in I$ s.t. $f(x_0) = m$; the proof that $\exists y_0 \in I$ s.t. $f(y_0) = M$ is similar.

$x_0 \in I$ st. $f(x_0) = m$ is similar: $\exists x_n \in I$ st. $f(x_n) < m + \frac{1}{n}$.

As above by BWT there is a convergent subseq $\{x_{n_k}\}_k$ of $\{x_n\}_n$. Then $x_0 = \lim x_{n_k} \in I$ and since $f$ is cts on $I$, $f(x_{n_k}) \rightarrow f(x_0)$.

We have: $m \leq f(x_{n_k}) < m + \frac{1}{n_k} \leq m + \frac{1}{k}$ for all $k$, so by the Squeeze Theorem $f(x_{n_k}) \rightarrow m$. Thus $f(x_0) = \inf_I f = \min_I f$. $\square$

**Note:** As in the above note, if $I$ is either not closed or not bdd, then $f$ need not assume a max. or min. value on $I$.

**Intermediate Value Theorem**

Sps that $f$ is cts on $I = [a,b]$. Then $\forall y$ between $f(a)$ and $f(b)$, $\exists x_0 \in I$ s.t. $y = f(x_0)$.

**Proof:** We may assume that $f(a) \neq f(b)$. Sps $f(a) < f(b)$ (the case $f(a) > f(b)$ is handled similarly), we may also assume $y \in (f(a), f(b))$.

Set $S = \{x \in I : y < f(x)\}$, then since $b \in S$, $S \neq \emptyset$. Put $x_0 = \inf_I S$.

Then since $a$ is a lower bd for $S$, $a \leq x_0 \leq b$.

For each $n$, $\exists x_n \in S$ s.t. $x_0 \leq x_n < x_0 + \frac{1}{n}$. Hence $x_n \rightarrow x_0$.

Since $x_n \in S$, $y < f(x_n)$ for all $n$ and hence $y \leq \lim f(x_n)$ but since $f$ is cts $f(x_0) = \lim f(x_n)$ so $y \leq f(x_0)$.

**Note:** that $a < x_0$ since $a \leq x_0$ but $f(a) < y \leq f(x_0)$ so $y = x_0 - a > 0$.

Set $z_n = x_0 - \frac{z}{n}$, then $a \leq z_n < x_0$ so $z_n \notin S$ and so $f(z_n) \leq y$.

Since $z_n \rightarrow x_0$ and $f$ is cts, $f(x_0) = \lim f(z_n) \leq y$. Combining the two inequalities (**), (***) we obtain $f(x_0) = y$. $\square$

**Cor:** Sps $f$ is cts on $I = [a,b]$. Then $f(I) = [m, M]$ where $m = \inf_I f$ and $M = \sup_I f$. (Note $m = M$ is possible.)
Cor: \( \text{Sp's } f \text{ is continuous on } I = [a, b] \text{ and } f(a)f(b) < 0, \)

Then \( \exists x_0 \in (a, b) \text{ s.t. } f(x_0) = 0. \)

Example: Show \( x = e^{-x} \) for some \( x \in [0, 1]. \) (We assume the function \( x \mapsto e^{-x} \) is cts.)

Def: Let \( I \) be an interval and let \( f : I \to \mathbb{R} \) be a function.

We say \( f \) is strictly incr. if \( \forall x, y \in I, x < y \Rightarrow f(x) < f(y) \) and strictly decr. if \( \forall x, y \in I, x < y \Rightarrow f(x) > f(y) \).

We say \( f \) is str. monotone if it is either str. incr. or str. decr.

Theorem: Let \( f : I \to \mathbb{R} \) be str. monotone where \( I \) is an interval.

and \( \text{Sp's that } f(I) \text{ is an interval. Then } f \text{ is cts.} \)

proof: We prove the assertion in the case \( I = [a, b] \) (the other cases are similar). Next \( \text{Sp's that } f \text{ is str. incr. Then} \)

\( f(I) = [f(a), f(b)]. \) Let \( c \in (a, b) \) and let \( \varepsilon > 0 \).

Since \( f(a) < f(c) < f(b) \), \( \exists u, v \text{ s.t.} \)

\( u \triangleq f(a) < u < f(c) < v < f(b). \)

\( \bigtriangleup u \leq \xi \leq v \text{ and } v - f(c) \leq \varepsilon. \)

\( 0 \leq \xi \leq \varepsilon \text{ and } v - f(c) \leq \varepsilon. \)

Now \( f(x) \leq f(c) + \varepsilon \text{ for all } x \in I. \text{ Hence } f \text{ is cts at } c. \text{ The case of endpoints } a, b \text{ is handled in a similar manner (but it is simpler).} \)

If \( f \) is str. decr., then \( -f \) is str. incr. so by the above \(-f \) is cts. Thus \( f \) itself is cts.

In Analysis, we loosen the definition of inverse a bit. If \( A \subseteq \mathbb{R} \) and \( f : A \to \mathbb{R} \text{ is } 1-1 \) and \( B = f(A), \) there is a unique function \( g : B \to \mathbb{R} \text{ s.t. } g(b) = A, \forall a \in A, g(f(a)) = a \text{ and } \forall b \in B, f(g(b)) = b. \)

We often write \( g = f^{-1} \text{ and call } g \text{ the inverse of } f. \)

**Proof:** Let \( I \) be an interval and let \( f : I \to \mathbb{R} \) be strictly monotone.

Then \( f \) is cts., \( J = f(I) \) is an interval and it has an inverse \( g : J \to \mathbb{R} \) which is strictly monotone and cts.