3.3 - Uniform Continuity
Recall: \( f : D \to \mathbb{R} \) is cts if \( \forall \epsilon > 0, \exists \delta > 0, \forall x, a \in D \)
\[ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon. \]

Note: Choice of \( \delta > 0 \) depends on both \( a \) and \( \epsilon > 0 \) in general.
Recall the example \( f(x) = x^2 \). Given \( a \in \mathbb{R}, \epsilon > 0 \) we took
\[ \delta = \min \left\{ 1, \frac{\epsilon}{2|a|+1} \right\} \]
so the bigger \( |a| \) is the smaller \( \delta > 0 \) must be.
Perhaps if we were to restrict \( f \) to a bounded set, we could choose \( \delta > 0 \) independently of \( a \). It is useful to have this stronger form of continuity.

Def: Let \( f \) be defined on \( D \). We say that \( f \) is uniformly continuous on \( D \) if \( \forall \epsilon > 0, \exists \delta > 0 \) s.t. \( \forall x, a \in D \)
\[ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon. \]

Notes: i) If \( f \) is unif. cts. on \( D \), then \( f \) is cts on \( D \).
ii) \( f \) is not unif. cts. on \( D \), iff
\[ \exists \epsilon > 0, \forall \delta > 0, \exists x, a \in D \text{ s.t. } |x - a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon. \]
iii) Equivalently, \( f \) is not unif. cts. on \( D \), iff
\[ \exists \epsilon > 0, \forall n \in \mathbb{N}, \exists x_n, a_n \in D \text{ s.t. } \forall n \text{ we have } |x_n - a_n| < \frac{1}{n} \text{ but } |f(x_n) - f(a_n)| > \epsilon. \]

Ex: We return to the case \( f(x) = x^2 \). We show first \( f \) is unif cts on \( D = [-M, M] \).
Let \( \epsilon > 0 \) and set \( \delta = \frac{\epsilon}{2M} \).

Then for all \( x, a \in D \),
\[ |x - a| < \delta \Rightarrow |x^2 - a^2| = |x + a||x - a| < (2M)\left(\frac{\epsilon}{2M}\right) = \epsilon. \]

But \( f \) is not unif cts on \( \mathbb{R} \). Set \( \epsilon = 1 \) and let \( \delta > 0 \),
Let \( n > \frac{1}{\delta} \) set \( x = n + \frac{1}{n} \) and \( a = n \). Then \( |x - a| = \frac{1}{n} < \delta \) and
\[ |x^2 - a^2| = \left((n+\frac{1}{n})^2 - n^2\right) = \frac{4}{n^2} < \frac{4}{n^2} < \epsilon. \]
Hence \( f \) is not unif. cts. on \( \mathbb{R} \).

Theorem: Let \( f \) be cts on \( I = [a, b] \). Then \( f \) is unif. cts. on \( I \).
Proof: Sps that \( f \) is not unif. cts. on \( I \). Then \( \exists \epsilon > 0 \) and \( \forall \delta > 0 \) \( \exists x_n, a_n \in I \) s.t.
\[ |x_n - a_n| < \delta \text{ but } |f(x_n) - f(a_n)| \geq \epsilon. \]
Since \( I \) is bdd it has a convergent subseq \( \{x_{n_k}\} \), say \( a_{n_k} \to a \). Observe that \( a \in I \).

We claim that \( x_{n_k} \to a \). Let \( \epsilon > 0 \), then \( \exists K \in I \text{ s.t. } \forall k > K \)
\[ |a_{n_k} - a| < \frac{\epsilon}{2} \text{ and } \forall k > K, \frac{1}{n_k} < \frac{1}{K} < \frac{\epsilon}{2}. \]

Thus for \( k > \max \{K, \frac{\epsilon}{2}\} \) we have
\[ |x_{n_k} - a| \leq |x_{n_k} - a_{n_k}| + |a_{n_k} - a| < \frac{1}{n_k} + \frac{\epsilon}{2} < \epsilon. \]
So \( x_{n_k} \to a \). Thus by the continuity of \( f \), \( \lim f(x_{n_k}) = f(a) \)
and \( \lim f(x_{n_k}) = f(a) \) and so \( \lim |f(a_{n_k}) - f(x_{n_k})| = 0 \).
But this contradicts the requirement \( |f(a_{n_k}) - f(x_{n_k})| \geq \epsilon \).
Hence \( f \) is unif. cts. on \( I \).
Example: Show that \( f(x) = \frac{1}{x^2} \) is unif cts. on \([\frac{1}{100}, 1000]\).

By the above Theorem it suffices to show that \( f \) is cts on \( I = [\frac{1}{100}, 1000] \) since \( I \) is closed and bdd. But \( f \) is a rational function and thus it is cts on its domain.

**Proposition:** Let \( f \) be unif cts on \( D \). If \( \{x_n\}_n \) is Cauchy in \( D \) then \( \{f(x_n)\}_n \) is also Cauchy.

**Proof:** Let \( \{x_n\}_n \) be a Cauchy seq in \( D \) and let \( \varepsilon > 0 \) be given. Then since \( f \) is unif cts on \( D \), \( \exists \delta > 0 \) s.t. \( \forall x, y \in D, \|x - y\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \). Since \( \{x_n\}_n \) is Cauchy \( \exists N \) s.t. for all \( m, n > N \), \( |x_m - x_n| < \delta \). Hence, \( |f(x_m) - f(x_n)| < \varepsilon \) for all \( m, n > N \).

Note: Given \( f: D \to \mathbb{R} \). If \( \exists \) Cauchy seq \( \{x_n\}_n \) in \( D \) s.t.

\( \{f(x_n)\}_n \) is not Cauchy, then \( f \) is not unif cts.

**Example:** Show \( f: (0, \infty) \to \mathbb{R} \) given by \( f(x) = \frac{1}{x} \) is not unif cts.

Set \( x_n = \frac{1}{n} \) for \( n \in \mathbb{N} \), then \( \{x_n\}_n \) is Cauchy in \( D = (0, \infty) \) (since it converges); but since \( f(x_n) = n \) \( \forall f(x_n) \}_n \) is not Cauchy. Hence \( f \) is not unif cts on \( D \).

Note: For any \( b > 0 \), \( f(x) = \frac{1}{x} \) is unif cts on \([b, \infty)\). Let \( \varepsilon > 0 \), then setting \( \delta = b^2 \varepsilon \), we have for all \( x, a \in [b, \infty) \) with \( |x - a| < \delta \),

\[ |f(x) - f(a)| = |\frac{1}{x} - \frac{1}{a}| = \frac{|x - a|}{|ax|} \leq \frac{|x - a|}{b^2} < \frac{\delta}{b^2} = \varepsilon. \]

So \( f \) is unif cts on \([b, \infty)\) as claimed.

**Proposition:** Let \( f \) be a cts fn on a bdd interval \( D = (a, b) \).

Then \( f \) can be extended to a cts function on \( \bar{D} = [a, b] \) iff \( f \) is unif cts on \( D \).

**Proof:** Follows immediately from the theorem.

Sketch of proof: \( \Rightarrow \) follows immediately from the theorem.

\( \Leftarrow \) Suppose \( f \) is unif cts on \( D \) and set \( \delta = \frac{b - a}{2} \) (note \( \delta > 0 \)).

Set \( x_n = a + \frac{\delta}{n} \). Then \( \{x_n\}_n \) is Cauchy in \( D \) and so by the above Proposition \( \{f(x_n)\}_n \) is also Cauchy and hence converges.

Similarly setting \( y_n = b - \frac{\delta}{n} \) for \( n \in \mathbb{N} \) yields another Cauchy seq. \( \{y_n\}_n \) in \( D \). So \( \{f(y_n)\}_n \) also converges.

We define \( \hat{f}: [a, b] \to \mathbb{R} \) by

\[ \hat{f}(x) = \begin{cases} f(x) & \text{if } a < x < b \\ \lim f(x_n) & \text{if } x = a \\ \lim f(y_n) & \text{if } x = b. \end{cases} \]

Since \( \hat{f} \) agrees with \( f \) on \( D \) it is cts at all pts in \( D \).

An \( \varepsilon/2 \) argument shows that \( \hat{f} \) is cts at each end pt.