3.4 Uniform Convergence

What does it mean for a seq. of fn's \( \{f_n\}_n \) defined on \( D \) to converge to a function \( f: D \to \mathbb{R} \)? The simplest notion is to require that \( f_n(x) \to f(x) \) for all \( x \in D \).

Examples.

(a) Define \( f_n: [0,1] \to \mathbb{R} \) by \( f_n(x) = x^n \). Then if we define \( f: [0,1] \to \mathbb{R} \) by \( f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases} \), we have \( f_n(x) \to f(x) \) for all \( x \in [0,1] \).

(b) Define \( g_n: [0,\infty) \to \mathbb{R} \) by \( g_n(x) = \frac{1}{n} 1_{x \leq \frac{1}{n}} 
\)

Then setting \( g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \), we have \( g_n(x) \to g(x) \) for all \( x \in [0,\infty) \).

(c) Define \( h_n: \mathbb{R} \to \mathbb{R} \) by \( h_n(x) = \begin{cases} -nx & x < 2/n \\ 0 & \text{otherwise} \end{cases} \).

Then \( h_n(x) \to 0 \) for all \( x \in \mathbb{R} \). To see this let \( \epsilon > 0 \).
then for all \( n > \frac{2}{\epsilon} \) we have \( h_n(x) = 0 \). If \( x \leq 0 \), then \( h_n(x) = 0 \) for all \( n \in \mathbb{N} \). So \( h_n(x) \to h(x) = 0 \) for all \( x \in \mathbb{R} \).

(d) Define \( k_n: [0,\infty) \to \mathbb{R} \) by \( k_n(x) = \frac{1}{x+n} \). Then for all \( x \in [0,\infty) \)

\( k_n(x) \to 0 \). To see this let \( \epsilon > 0 \). Then for \( n > \frac{1}{\epsilon} \) we have

\[ |k_n(x) - 0| = \left| \frac{1}{x+n} \right| < \frac{1}{n} < \epsilon \] 

Note that the limit functions in (a) and (b) are not cts even though the functions in the seqs. are.

Def: Let \( D \subseteq \mathbb{R} \), let \( \{f_n\}_n \) be a seq of fn's defined on \( D \) and let \( f \) be a fn defined on \( D \).

(a) Say \( \{f_n\}_n \) converges ptwise to \( f \) if \( \forall x \in D, \exists \eta \in \mathbb{N} \) st. \( \forall n \geq \eta, |f_n(x) - f(x)| < \epsilon \).

(b) Say \( \{f_n\}_n \) converges uniformly to \( f \) if \( \forall \epsilon > 0, \exists \eta \in \mathbb{N} \) st. \( \forall x \in D, \forall n \geq \eta, |f_n(x) - f(x)| < \epsilon \).

Notes: a) If \( f_n \to f \) uniformly, then \( f_n \to f \) pointwise.

b) \( \{f_n\}_n \) does not converge uniformly to \( f \) if \( \exists \epsilon > 0, \forall N \in \mathbb{N}, \exists x \in D, \forall n \geq N, |f_n(x) - f(x)| \geq \epsilon \).

c) In all four examples above convergence is pointwise but the convergence is uniform only in (d).

to see convergence is not uniform in (c). Set \( \epsilon = \frac{1}{2} \) and let \( N \) be given. Let \( n > N \) and set \( x = \frac{1}{2n} \). Then \( h_n(x) = h(\frac{1}{2n}) = \frac{1}{2} \).

So \( |h_n(x) - h(x)| = |\frac{1}{2} - 0| = \frac{1}{2} \geq \epsilon \).

To see that \( k_n \) is unif. on \( D \). Let \( \epsilon > 0 \), then for \( n > \frac{1}{\epsilon} \)

we have \( |k_n(x) - 0| = \left| \frac{1}{x+n} \right| < \frac{1}{n} < \epsilon \) for all \( x \in [0,\infty) \).

So \( k_n \to 0 \) unif. on \( D \).
Theorem: Let \( \{ f_n \}_{n=1}^{\infty} \) be a seq. of cts fns def on \( D \) and Sps that \( f_n \to f \) unit to \( f: D \to \mathbb{R} \). Then \( f \) is cts.

Proof: Let \( x_0 \in D \) and let \( \varepsilon > 0 \). Since \( f_n \to f \) unit on \( D \), there exists an N s.t. \( \forall x \in D, n > N, |f_n(x) - f(x)| < \frac{\varepsilon}{3} \). Let \( m > N \). Then since \( f_n \) is cts, \( \exists \delta > 0 \) s.t. \( \forall x \in D, |x - x_0| < \delta \Rightarrow |f_n(x) - f(x_0)| < \frac{\varepsilon}{3} \).

Then for \( x \in D \) s.t. \( |x - x_0| < \delta \) we have

\[
|f(x) - f(x_0)| 
\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Hence, \( f \) is cts at \( x_0 \). Since \( x_0 \) was chosen arbitrarily, \( f \) is cts.

Note: It follows by this theorem that convergence is not uniform in examples (a), (b) above since the limit functions are not continuous.

Prop: Let \( \{ f_n \}_{n=1}^{\infty} \) be a seq. of fns defined on \( D \) and let

\( f: D \to \mathbb{R} \) be given.

a) Sps \( \{ f_n \}_{n=1}^{\infty} \) in \( \mathbb{R} \) s.t. \( b_n \to 0 \) ad \( |f_n(x) - f(x)| \leq b_n \) for all \( x \in D, n \in \mathbb{N} \).

Then \( f_n \to f \) unit on \( D \).

b) \( f_n \to f \) unit on \( D \) iff \( \sup |f_n - f| \to 0 \)

Proof of (a): Let \( \varepsilon > 0 \). Then since \( b_n \to 0 \), \( \exists N \) s.t. \( b_n < \frac{\varepsilon}{3} \) for all \( n > N \).

Then for all \( x \in D \) and \( n > N \) we have \( |f_n(x) - f(x)| < b_n < \varepsilon \).

Hence \( f_n \to f \) unit.

Ex: This gives an easy proof that the convergence \( f_n \to f \) above is uniform by taking \( b_n = \frac{1}{n} \).

Def: Let \( \{ f_n \}_{n=1}^{\infty} \) be a seq. of fns def on \( D \subset \mathbb{R} \). We say that \( \{ f_n \}_{n=1}^{\infty} \) is unit Cauchy on \( D \) if \( \forall \varepsilon > 0, \exists N \) s.t.

\( \forall x \in D, \forall m, n > N, |f_m(x) - f_n(x)| < \varepsilon \).

Prop: Let \( \{ f_n \}_{n=1}^{\infty} \) be a seq. of fns def on \( D \subset \mathbb{R} \). Then \( \{ f_n \}_{n=1}^{\infty} \) is unit Cauchy iff \( \{ f_n \}_{n=1}^{\infty} \) converges unit to \( f \) on \( D \).