4.1 - Limits of Functions

To define the derivative we need to use the concept of limit at a pt, where the function may not be defined.

An open interval is any interval of form \((a, b), (-\infty, b), (a, \infty), \mathbb{R}\).

\textbf{Def: } Let \(I\) be an open interval and let \(a \in I\). Say that \(L\) is the limit of \(f\) at \(a\) if \(\forall \varepsilon > 0, \exists \delta > 0\) s.t. \(\forall x \in I, 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\).

In this case we write \(\lim_{x \to a} f(x) = L\).

\textbf{Notes: } i) The limit is unique if it exists.

ii) \(\forall x \in I, a \neq I; \) \(f\) is \(ct\) at \(a\) iff \(\lim_{x \to a} f(x) = f(a)\).

iii) If \(f(x) = c\) for all \(x \in I, a \neq I\), then \(\lim_{x \to a} f(x) = c\).

iv) \(\forall \varepsilon > 0, \exists \delta > 0\) s.t. \(0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\).

\textbf{Ex: } Show that \(\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = 3\).

Set \(f(x) = \frac{x^2 - x - 2}{x - 2}\), then \(f(x) = x + 1\) for \(x \neq 2\). Hence, \(\lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 1) = 3\) since \(g(x) = x + 1\) is \(ct\) at \(2\).

One-sided limits & limits at \(\pm \infty\)

We may treat limits at \(\pm \infty\) as 1-sided limits (using diff. terminology).

\textbf{Def: } \(Sps\ f\) is defined on \(I = (a, b)\) (allow \(a = -\infty\) or \(b = \infty\)). We say

Let \(I\) is the \(right\) limit of \(f\) at \(a\) (or \(limit\ of \(f^+\) at \(a\)) and write:

\[ L = \lim_{x \to a^+} f(x) \]

\(\forall \varepsilon > 0, \exists \delta > 0\) s.t. \(\forall \varepsilon > 0\) s.t. \(\forall x \in I, a < x < m => |f(x) - L| < \varepsilon\).

Similarly, \(L\) is the \(left\) limit of \(f\) at \(b\) (or \(limit\ of \(f^-\) at \(\infty\)) and write:

\[ L = \lim_{x \to b^-} f(x) \]

\(\forall \varepsilon > 0, \exists \delta > 0\) s.t. \(\forall x \in I, m < x < b => |f(x) - L| < \varepsilon\).

\textbf{Ex: } Show that \(\lim_{x \to \infty} \frac{2x}{x + 1} = 2\).

Let \(\varepsilon > 0\). Set \(m = 2/\varepsilon\) and let \(x > \frac{2}{\varepsilon}\). Then

\[ |\frac{2x}{x + 1} - 2| = |\frac{2x - 2(x + 1)}{x + 1}| = \frac{2}{x + 1} < \frac{2}{x} < \varepsilon.\]

\textbf{Prop: } Let \(I = (a, b)\), open interval, let \(a \in I\), let \(L \in \mathbb{R}\) and \(Sps\ f\)

is def. on \(I \setminus \{a\}\). Then

\[ \lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).\]

\textbf{proof: } \(\Rightarrow \) "easy"

\(\Leftarrow \) \(Sps\ \) both 1-sided limits exist and equal \(L\). Let \(\varepsilon > 0\).

Then \(\exists \delta > 0\) s.t. \(\forall x \in I, a < x < a + \delta \Rightarrow |f(x) - f(a^-)| < \varepsilon\).

Hence, \(\forall x \in I, 0 < |x - a| < \delta \Rightarrow |f(x) - f(a^-)| < \varepsilon, \ thus \ \lim_{x \to a} f(x) = L.\)
4.1 - Limits of functions - continued

Ex. Let \( f(x) = \begin{cases} x^2 + 2 & x < 1 \\ 4 - x & x \geq 1 \end{cases} \). Use above Prop. to show \( \lim_{x \to 1} f(x) = 3 \).

**Theorem:** Let \( I \) be an open interval, \( \alpha \in I \), \( f \) is defined on \( I \setminus \{ \alpha \} \). Then \( \lim_{x \to \alpha} f(x) = L \) iff \( \forall \epsilon > 0 \) there is \( \delta > 0 \) such that if \( 0 < |x - \alpha| < \delta \), then \( |f(x) - L| < \epsilon \).

A similar assertion holds for 1-sided limits and limits at \( \pm \infty \). The proof is similar to the proof of the sequential character of continuity.

Ex. Show that \( \lim_{x \to 0} \cos x \) does not exist.

Suppose that it does exist and \( L = \lim_{x \to 0} \cos x \). Set \( a_n = \pi n \), then since \( a_n \to \infty \) it follows by the theorem that \( L = \lim_{n \to \infty} \cos n\pi = \lim_{n \to \infty} \cos (-1)^n \). But since \( \cos (-1)^n \) does not converge as \( n \to \infty \) we get a contradiction. So \( \lim_{x \to 0} \cos x \) does not exist.

**Main Limit Theorem for Functions**

Let \( a \in I \), where \( I \) is an open interval, \( f, g \) def on \( I \setminus \{ a \} \) and \( s_p_s \) that \( L = \lim_{x \to a} f(x) \) and \( M = \lim_{x \to a} g(x) \). Then

i) For \( c, d \in \mathbb{R} \), \( \lim_{x \to a} (c f(x) + d g(x)) = c L + d M \).

ii) \( \lim_{x \to a} f(x) g(x) = L M \).

iii) If \( M \neq 0 \), there is an open interval \( J \ni a \) s.t. \( g(x) \neq 0 \) for \( x \in J \setminus \{ a \} \) and \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \).

iv) Let \( h \) be cts on an open interval cont \( L \) and \( f(I) \). Then \( \lim_{x \to a} h(f(x)) = h(L) \).

Similar results hold for 1-sided limits and limits at \( \pm \infty \). The proofs are quite similar to the analogous proofs for cts fn's with the above theorem used in place of the sequential characterization of continuity.

**Squeeze Theorem:** Let \( a \in I \), an open interval, and let \( f, g, h \) be defined on \( I \setminus \{ a \} \). Sps that

i) \( \forall x \in I \setminus \{ a \}, f(x) \leq g(x) \leq h(x) \)

ii) \( \lim_{x \to a} f(x) = L = \lim_{x \to a} g(x) \).

Then \( \lim_{x \to a} g(x) = L \). Similar results hold for 1-sided limits and limits at \( \pm \infty \).

Ex. Prove \( \lim_{x \to 0} x \sin(\frac{1}{x}) = 0 \).

Observe that \(-1 \leq x \sin(\frac{1}{x}) \leq x \) and \( \lim_{x \to \infty} x \to \infty \).

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4.1 - Limits of Functions - continued.

Infinite Limits:
Let $a \in I$, open interval, and suppose that $f$ is defined on $I \setminus \{a\}$.

**Def:** (i) $\lim_{x \to a^-} f(x) = \infty$, if $\forall M > 0$, $\exists \delta > 0$ s.t. $x \in I$, $0 < |x-a| < \delta \implies f(x) > M$.

(ii) $\lim_{x \to a^-} f(x) = -\infty$, if $\forall M < 0$, $\exists \delta > 0$ s.t. $x \in I$, $0 < |x-a| < \delta \implies f(x) < M$.

Similar definition hold for 1-sided limits and (limits at $\pm \infty$).

**Note:** $\lim_{x \to a^-} f(x) = \infty$ iff $\lim_{x \to a^+} f(x) = \infty = \lim_{x \to a^-} f(x)$.

**Ex:** Show that $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^+} \frac{1}{x} = -\infty$.

**Prop:** With $I, a, f$ as above, $\lim_{x \to a^-} f(x) = \infty$ iff $\forall n \in \mathbb{N}, \exists \delta > 0$ s.t. $a_n \in I \setminus \{a\}, a_n \to a \implies f(a_n) \to \infty$.

**proof:** $\iff$ straightforward.

(contrapositive) Suppose it is not the case that $\lim_{x \to a^-} f(x) = \infty$.

Then $\exists M > 0$ s.t. $\forall \delta > 0$, $\exists x \in I$ s.t. $0 < |x-a| < \delta$ and $|f(x)| \leq M$.

Hence, for each $n$, $\exists a_n \in I$ s.t. $0 < |a_n-a| < \frac{1}{n}$ and $f(a_n) \leq M$.

Thus $a_n \to a$ but (contrapositive) $\lim_{x \to a^-} f(x) \not\to \infty$.

There is a similar characterization for $\lim_{x \to a^-} f(x) = -\infty$ and for 1-sided limits and limits at $\pm \infty$.

See also Theorem 4.1.15 which is similar to 2.4.7a for seqs.

**Prop:** Suppose $f$ is defined on $I = (a, b)$.

(i) If $f(x) > 0$ for all $x \in I$, then $\lim_{x \to a^+} f(x) = 0$ iff $\lim_{x \to a^-} \frac{1}{f(x)} = 0$.

(ii) If $f(x) < 0$ for all $x \in I$, then $\lim_{x \to a^+} f(x) = -\infty$ iff $\lim_{x \to a^-} \frac{1}{f(x)} = 0$.

Similar results hold for left limits, 2-sided limits and limits at $\pm \infty$. 

[Image]