4.4 - L'Hôpital's Rule

This is a useful tool for evaluating limits of indeterminate forms.

Cauchy's Mean Value Theorem

If $f, g$ are cts on $[a, b]$ and diff on $(a, b)$, then $\exists \ c \in (a, b)$ s.t.
\[
(f(b)-f(a))(g'(c)) = (g(b)-g(a))(f'(c)).
\]
Moreover, if $g'(x) \neq 0$ for all $x \in (a, b)$, $g(b) \neq g(a)$ & $\exists \ c \in (a, b)$ s.t.
\[
\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.
\]

Proof: Let $h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$ for $x \in [a, b]$. Then $h$ is cts on $[a, b]$ and diff on $(a, b)$ (because $h$ is a linear combination of $f$ and $g$ which satisfy both properties). A routine calculation yields $h'(x) = 0$ for some $c \in (a, b)$ and so $(f(b)-f(a))g'(c) - (g(b)-g(a))f'(c) = 0$. Hence, eq. (**) above holds.

Now suppose that $g'(x) \neq 0$ for all $x \in (a, b)$. Then by Rolle's Theorem $g(b) \neq g(a)$. Then dividing both sides of $h'(c)$ by $g'(c)(g(b)-g(a))$ yields $h'(c) = 0$.

L'Hôpital's Rule for limits of type $"0/0"

Suppose that $f, g$ are cts on $I = (a, b)$ (possibly infinite interval).
Let $u = a^+$, $b^-$ or let $u \in I$. Suppose that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in I \setminus \{u\}$, and that
\[
\lim_{x \to u} f'(x) = L, \quad \lim_{x \to u} g'(x) = M.
\]
\[\text{i}) \quad \text{If} \ \lim_{x \to u} \frac{f(x)}{g(x)} = L < \infty, \ \text{then} \ \lim_{x \to u} \frac{f(x)}{g(x)} = L.
\]
\[\text{ii}) \quad \text{If} \ \lim_{x \to u} \frac{f(x)}{g(x)} = \pm \infty, \ \text{then} \ \lim_{x \to u} \frac{f(x)}{g(x)} = \pm \infty.
\]

Proof: (i) We prove this in the case that $u = a^+$, the other cases are similar.

Suppose that $\lim_{x \to a^+} f(x) = L \in \mathbb{R}$. Let $\epsilon > 0$. Then $\exists \ e \in (a, b)$ s.t.

\[
\left| \frac{f(x)}{g(x)} - L \right| < \frac{\epsilon}{2}
\]
whenever $a < x < m$. Now let $x, y \in (a, e)$ s.t. $x > y$. Then by Cauchy Mean Value Theorem (CMVT), $\exists \ c \in (y, x)$ s.t.

\[
\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}.
\]
Since $c \in (a, e)$, we have

\[
\left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \frac{\epsilon}{2}.
\]
Since $\lim_{y \to a^+} f(y) = 0$ and $\lim_{y \to a^+} g(y) = 0$, we have

\[
\left| \frac{f'(c)}{g'(c)} - L \right| \leq \frac{\epsilon}{2}.
\]
Hence, $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

(ii) We only consider the case $u = a^+$. So that $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty$.

Let $M > 0$. Then $\exists \ e \in (a, b)$ s.t. $\frac{f(x)}{g(x)} > 2M$ for all $x \in (a, e)$.

Let $x, y \in (a, e)$ with $y < x$. Then $\exists \ c \in (y, x)$ s.t.

\[
\frac{f(x) - f(y)}{x - y} = \frac{f'(c)}{g'(c)}.
\]
4.4 - L'Hôpital's Rule ctd.

Since \( c \in (a, m) \) we have

\[
\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)} > 2M.
\]

Taking the limit as \( y \to a^+ \) we have \( \frac{f(x)}{g(x)} > 2M > M \).

Hence \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty \).

L'Hôpital's Rule for limits of type "\( \frac{\infty}{\infty} \)"

Suppose \( f, g \) are differentiable on \( I = (a, b) \) (possibly infinite interval).

Let \( a = a^+, b^- \) or let \( a \in I \). Supposed \( g(x) \neq 0 \) and \( g'(x) \neq 0 \) for all \( x \in I \) and that \( \lim_{x \to a^+} f(x) = \pm \infty = \lim_{x \to a^-} g(x) \). Then

i) If \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = L \in \mathbb{R} \), then \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = L \).

ii) If \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = \pm \infty \), then \( \lim_{x \to a^+} \frac{f(x)}{g(x)} = \pm \infty \).

In the following example assume the basic properties of the elementary functions.

Ex. Show that \( \lim_{x \to 0^+} x \ln x = 0 \).

We write \( x \ln x = \frac{\ln x}{\frac{1}{x}} \) observe that \( f(x) = \ln x \), \( g(x) = \frac{1}{x} \)

are both differentiable on \( I = (0, \infty) \) and that for all \( x \in I \),

\( g(x) = \frac{1}{x} \neq 0 \) and \( g'(x) = -\frac{1}{x^2} \neq 0 \). Moreover,

\[ \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \ln x = -\infty \text{ and } \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty. \]

We have \( \lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0 \).

Hence by L'Hôpital's Rule for limits of type "\( \frac{\infty}{\infty} \)" we have

\[ \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{f(x)}{g(x)} = 0. \]