1. Prove that
\[ \lim_{n \to \infty} \frac{\cos n^2}{\sqrt{n}} = 0. \]

We apply Theorem 2.3.2 with \( a_n = 1/\sqrt{n} \) and \( b_n = \cos n^2 \). Since \( |b_n| = |\cos n^2| \leq 1 \) for all \( n \), \( \{b_n\}_n \) is bounded. Moreover, \( a_n = 1/\sqrt{n} \to 0 \) (by Theorem 2.3.6(f)). Therefore by Theorem 2.3.2, \( \frac{\cos n^2}{\sqrt{n}} = a_n b_n \to 0. \)

2. Let \( \{a_n\}_n \) and \( \{b_n\}_n \) be sequences of positive numbers. Suppose that \( a_n \to \infty \) and that there exist positive constants \( N, \varepsilon \) such that \( b_n \geq \varepsilon \) for all \( n > N \). Prove that \( a_n b_n \to \infty \). (Please don’t use Theorem 2.4.7(e).)

Let \( M > 0 \). Then since \( a_n \to \infty \), there is \( N_1 \) such that \( a_n > M/\varepsilon \) for all \( n > N_1 \). Hence for all \( n > N_2 = \max\{N, N_1\} \) we have \( b_n \geq \varepsilon \) and \( a_n > M/\varepsilon \). Therefore
\[ a_n b_n \geq a_n \varepsilon > \frac{M}{\varepsilon} \varepsilon = M. \]

for all \( n > N_2 \) and so \( a_n b_n \to \infty. \)

3. Let \( \{a_n\}_n \) be a sequence of nonnegative numbers and let
\[ s_n = \sum_{k=1}^{n} a_k \text{ for } n \geq 1. \]

Prove that either \( \{s_n\} \) converges or \( s_n \to \infty. \)

Since \( a_k \geq 0 \) for all \( k \), we have \( s_{n+1} = s_n + a_{n+1} \geq s_n \) for all \( n \). Hence, \( \{s_n\} \) is nondecreasing. Therefore by the Monotone Convergence Theorem, either \( \{s_n\} \) is bounded above and thus converges or it is not and so \( s_n \to \infty. \)

4. Prove that
\[ \lim_{n \to \infty} \frac{4n^2 + 7}{3n^2 + 5} = \infty. \]

A straightforward calculation shows that
\[ \frac{4n^2 + 7}{3n^2 + 5} = n \left( \frac{4 + 7/n^2}{3 + 5/n} \right). \]

Observe that \( a_n = n \to \infty \) and \( b_n = \frac{4 + 7/n^2}{3 + 5/n} \to \frac{4}{3} \) by the Main Limit Theorem (Theorem 2.3.6(a), (b), (d)). Set \( \varepsilon = 1 \); then since \( b_n \to 4/3 > \varepsilon \) there is \( N \) such that \( b_n \geq \varepsilon \) for all \( n > N \) by Theorem 2.2.3. Hence, by Exercise 2, \( a_n b_n \to \infty \) and so the desired result holds. (Note that one can also use the proposition proved in class that asserts: if \( a_n \to \infty \) and \( b_n \to b > 0 \), then \( a_n b_n \to \infty \).)

5. Let \( \{a_n\}_n \) be a sequence defined recursively by \( a_1 := 1 \) and \( a_{n+1} := \frac{1}{4}(2a_n + 3) \) for \( n \in \mathbb{N}. \)

a. Show that \( a_n \leq 2 \) for all \( n \in \mathbb{N}. \)

We prove this by induction. For the base case observe that \( a_1 = 1 \leq 2. \) Next suppose that \( a_n \leq 2 \) for some \( n \in \mathbb{N}. \) Then since \( 2a_n + 3 \leq 7 \) we have
\[ a_{n+1} = \frac{2a_n + 3}{4} \leq \frac{7}{4} \leq 2. \]

Hence, the desired result follows by induction.

b. Show that \( a_n \leq a_{n+1} \) for all \( n \in \mathbb{N}. \)

We prove this by induction. We have \( a_2 = 5/4 \geq 1 = a_1 \) and so the base case holds. Next suppose that \( a_n \leq a_{n+1} \) for some \( n \in \mathbb{N}. \) Then \( 2a_n + 3 \leq 2a_{n+1} + 3 \) and so we have
\[ a_{n+1} = \frac{2a_n + 3}{4} \leq \frac{2a_{n+1} + 3}{4} = a_{n+2}. \]

Hence, by induction we have \( a_n \leq a_{n+1} \) for all \( n \in \mathbb{N}. \)

5. Prove that \( \{a_n\}_n \) converges and find the limit.

By part (a) \( \{a_n\}_n \) is bounded above and by part (b) \( \{a_n\}_n \) is nondecreasing. Therefore \( \{a_n\}_n \) converges by the Monotone Convergence Theorem (Theorem 2.4.1). Let \( a \) denote the limit of \( \{a_n\}_n \). Then since \( a_{n+1} \to a \) we have by the Main Limit Theorem (Theorem 2.3.6(a), (b))
\[ a = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{4}(2a_n + 3) = \frac{1}{4} \left( 2 \lim_{n \to \infty} a_n + \lim_{n \to \infty} 3 \right) = \frac{1}{4}(2a + 3). \]

Hence, \( a = (2a + 3)/4 \) and so the limit is \( a = 3/2. \)