1. Let \( \{a_n\}_n \) be a sequence in \( \mathbb{R} \) and suppose that \( \limsup a_n = \infty \). Prove that there is a subsequence \( \{a_{n_k}\}_k \) of \( \{a_n\}_n \) such that \( a_{n_k} \to \infty \). Do not use Theorem 2.6.5.

We first observe that \( \{a_n\}_n \) is not bounded above and so \( s_n = \sup\{a_k : k \geq n\} = \infty \) for all \( n \in \mathbb{N} \). We construct a subsequence \( \{a_{n_k}\}_k \) of \( \{a_n\}_n \) recursively. Since \( s_1 = \infty \), there is an \( n_1 \geq 1 \) such that \( a_{n_1} > 1 \). Now suppose that \( n_1, \ldots, n_k \in \mathbb{N} \) have been chosen such that \( n_1 < \cdots < n_k \) and \( a_{n_j} > j \) for \( j = 1, \ldots, k \). Then since \( s_{n_k+1} = \infty \), there is a natural number \( n_{k+1} \geq n_k + 1 \) such that \( a_{n_{k+1}} > k + 1 \). This yields a subsequence \( \{a_{n_k}\}_k \) of \( \{a_n\}_n \) such that \( a_{n_k} > k \) for all \( k \in \mathbb{N} \). We now check that \( a_{n_k} \to \infty \). Let \( M > 0 \); then for all \( k > M \) we have \( a_{n_k} > k > M \). Hence \( a_{n_k} \to \infty \) as required.

2. Let \( f \) be given by \( f(x) := x^3 \) for all \( x \in \mathbb{R} \). Use the definition to prove that \( f \) is continuous at 2.

If \( |x - 2| \leq 1 \), then \( |x| \leq |x - 2| + 2 \leq 3 \) and

\[
|x^2 + 2x + 4| \leq |x^2| + |2x| + 4 \leq 9 + 6 + 4 = 19.
\]

Let \( \varepsilon > 0 \) and set \( \delta := \min\{1, \varepsilon/19\} \). Let \( x \in \mathbb{R} \) and suppose that \( |x - 2| < \delta \). Then the above inequality holds and we have

\[
|f(x) - f(2)| = |x^3 - 8| = |x^2 + 2x + 4||x - 2| \leq 19|x - 2| < 19\delta \leq \varepsilon.
\]

Hence, \( f \) is continuous at 2.

3. Use the sequential characterization of continuity (Theorem 3.1.6) to prove Theorem 3.1.11.

Let \( f, g \) and \( a \in D_{f \circ g} \) be as in the statement of Theorem 3.1.11. Suppose that \( g \) is continuous at \( a \) and that \( f \) is continuous at \( g(a) \). Let \( \{a_n\}_n \) be a sequence in \( D_{f \circ g} \) that converges to \( a \). Since \( g \) is continuous at \( a \) and \( \{a_n\}_n \) is a sequence in \( D_g \) that converges to \( a \), \( \{g(a_n)\}_n \) converges to \( g(a) \) by the sequential characterization of continuity (Theorem 3.1.6). Moreover, since \( f \) is continuous at \( g(a) \) and \( \{g(a_n)\}_n \) is a sequence in \( D_f \) that converges to \( g(a) \), \( \{f(g(a_n))\}_n \) converges to \( f(g(a)) \) by the sequential characterization of continuity (Theorem 3.1.6). Hence, \( \{(f \circ g)(a_n)\}_n \) converges to \( (f \circ g)(a) \). Therefore, \( f \circ g \) is continuous at \( a \) by the sequential characterization of continuity (Theorem 3.1.6).

4. Prove that the function \( g \) defined below is not continuous at 0 but is continuous everywhere else. (You may assume that the cosine function is continuous.)

\[
g(x) := \begin{cases} 
\cos \frac{1}{x} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

We use the sequential characterization of continuity (Theorem 3.1.6) to prove that \( f \) is not continuous at 0 by finding a sequence \( \{a_n\}_n \) such that \( a_n \to 0 \) but \( f(a_n) \neq f(0) \). Set \( a_n := 1/2n\pi \). Then \( a_n \to 0 \) but

\[
\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \cos 2n\pi = 1 \neq 0 = f(0);
\]

hence, \( f(a_n) \neq f(0) \) and so \( f \) is not continuous at 0 by Theorem 3.1.5. To prove that \( f \) is continuous elsewhere it suffices to prove that its restriction to \( U = \mathbb{R} \setminus \{0\} \) is continuous. Since the restriction to \( U \) is a composition of two continuous functions (namely, \( x \mapsto 1/x \) and \( x \mapsto \cos x \)), it is continuous by Theorem 3.1.11.

5. Let \( f \) be a continuous function with domain \( D_f = [a, b] \) and suppose that \( f(a) < f(b) < f(c) \) for some \( c \in (a, b) \). Prove that \( f \) is not one-to-one.

Since \( f \) is continuous on \([a, b]\) it is continuous on the subinterval \([a, c]\). Since \( f(b) \) lies between \( f(a) \) and \( f(b) \) we may apply the Intermediate Value Theorem (Theorem 3.2.3) with \( y = f(b) \) to conclude that there is some \( d \in (a, c) \) such that \( f(d) = f(b) \). Hence, \( f \) is not one-to-one.