1. Let \( x \) be a real number with \( x \neq 1 \). Show that for all \( n \in \mathbb{N} \)
\[
1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.
\]
2. Prove that \( 1^2 + 3^2 + \cdots + (2n - 1)^2 = \frac{(4n^3 - n)}{3} \) for all \( n \in \mathbb{N} \).
3. Find the set of all natural numbers \( n \) for which \( 2^n < 2^n \). Prove your answer is correct (induction may help).
4. Let \( S \) be a bounded nonempty set of real numbers and let \( a \in \mathbb{R} \).
   - a. Define \( a + S := \{ a + s : s \in S \} \). Prove that \( \sup(a + S) = a + \sup S \) and \( \inf(a + S) = a + \inf S \).
   - b. Suppose \( a \neq 0 \) and define \( aS := \{ ax : s \in S \} \). Prove that
     \[
     \sup aS = \begin{cases} 
     a \sup S & \text{if } a > 0, \\
     a \inf S & \text{if } a < 0.
     \end{cases}
     \]
5. With \( f \) and \( A \) as given, find \( \sup_A f \) and \( \inf_A f \). Do \( \max_A f \) or \( \min_A f \) exist? If so find them.
6. Let \( a, b \in \mathbb{R} \) and suppose that for every \( \varepsilon > 0 \), \( a \leq b + \varepsilon \). Prove that \( a \leq b \).
7. Let \( S := (1, \infty) \cap \mathbb{Q} \). Prove that \( \sup S = \infty \) and \( \inf S = 1 \).
8. Suppose that \( S \) is a nonempty subset of \( \mathbb{R} \). Show that if there is an element \( u \in S \) which is an upper bound of \( S \), then \( u = \sup S \).
9. Let \( S := \{ 1 + \left( \frac{-1}{n+1} \right) : n \in \mathbb{N} \} \). Show that \( S \) is bounded and find \( \sup S \) and \( \inf S \).
10. Let \( S := \{ \frac{1}{n+1} - \frac{1}{n} : m, n \in \mathbb{N} \} \). Show that \( S \) is bounded and find \( \sup S \) and \( \inf S \).
11. Let \( x \in \mathbb{R} \) be given. Show that for every \( \varepsilon > 0 \) there is a rational number \( r \) such that \( |r - x| < \varepsilon \). Use this fact to show that there is a sequence of rational numbers which converges to \( x \).
12. Use the definition of the limit of a sequence to establish the following limit.
\[
\lim_{n \to \infty} \frac{3n - 4}{2n + 1} = \frac{3}{2}.
\]
13. Show that the sequence \( \{ \sqrt{n^2 + 2n} - n \} \) converges and find its limit (cite any theorems you use).
14. Show that the sequence \( \{ \ln(n^2 + 1) \} \) diverges to \( \infty \).
15. Let \( \{ a_n \} \) be a sequence of positive numbers. Use the definition to prove that if \( a_n \to \infty \), then \( \sqrt{a_n} \to \infty \).
16. Let \( \{ a_n \} \) be a sequence of real numbers and let \( m \in \mathbb{N} \). Define the sequence \( \{ b_n \} \) by \( b_n := a_{m+n} \). Prove that \( \{ a_n \} \) converges iff \( \{ b_n \} \) does. Prove that if either (and so both) converges, then they have the same limit.
17. Prove that
\[
\lim_{n \to \infty} \frac{3n^2 - 4}{2 - 3n} = -\infty.
\]
18. Use limit theorems (clearly cite the ones you use) to prove the sequence \( \{ x_n \} \) converges (and find its limit):
\[
x_n := \frac{n^2 - 5}{2n^2 + 3n + 4}.
\]
19. Suppose \( \{ a_n \} \) is a sequence of positive real numbers and that there is a number \( r \in (0, 1) \) such that \( a_{n+1} < ra_n^r \) for all \( n \in \mathbb{N} \). Use induction to show that \( a_{n+1} < a_1 r^n \) for all \( n \in \mathbb{N} \). Show that
\[
\lim_{n \to \infty} a_n = 0.
\]
20. Show that the sequence \( \{ a_n \} \) defined below converges and find its limit:
\[
a_n := \frac{n(2 + \cos n)}{n^2 + 1}.
\]
21. Let \( \{ a_n \} \) be a sequence defined recursively by \( a_1 := 0 \) and \( a_{n+1} = \sqrt{1 + a_n} \) for all \( n \in \mathbb{N} \). Prove that \( \{ a_n \} \) converges and find its limit.
22. Let \( \{ a_n \} \) be a sequence of nonnegative real numbers and let \( \{ b_n \} \) be a sequence such that \( b_{n+1} - b_n = a_n \) for all \( n \in \mathbb{N} \). Prove that \( \{ b_n \} \) converges if it is bounded above and diverges to \( \infty \) otherwise.
23. Suppose that \( a_n \to \infty \) and that \( \{ b_n \} \) is bounded above. Use the definition to prove that \( b_n - a_n \to -\infty \).
24. Suppose that \( \{ a_n \} \) is not bounded above. Prove that for every \( M, N > 0 \), there is \( n > N \) such that \( a_n > M \).
25. Use the definition to prove that \( \frac{1}{2}(2n - 5) \to \infty \).
26. Let \( \{ a_n \} \) be defined recursively by \( a_1 := 1 \) and \( a_{n+1} = a_n + 1/a_n \) for all \( n \in \mathbb{N} \). Does \( \{ a_n \} \) converge?
27. Let \( \{ a_n \} \) be defined recursively by \( a_1 := 7 \) and \( a_{n+1} = 2 - 1/a_n \) for all \( n \in \mathbb{N} \). Does \( \{ a_n \} \) converge?