10.1 - Integration over a Rectangle

10.1a - Def: A subset $R \subset \mathbb{R}^d$ is called a rectangle if $\exists a_i, b_i \in \mathbb{R}$ st. $a_i \leq b_i$ for $(i = 1, \ldots, d)$ s.t.

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d] = \{ x \in \mathbb{R}^d : a_i \leq x_i \leq b_i, i = 1, \ldots, d \}$$

(say $R$ degenerate). Define $V(R) = (b_1 - a_1) \cdots (b_d - a_d)$; we refer to $V(R)$ as the volume of $R$.

10.1b - Notation: A partition $P$ of a rectangle $R$ as above is given by partitioning each $[a_k, b_k]$ for $k = 1, \ldots, d$; $P_k = \{ x_0, x_1, x_2, \ldots, x_{n_k} = b_k \}$; $P_k^i$.

This partition subdivides $R$ into subrectangles $I_{j_1}^1 \times \cdots \times I_{j_d}^1$, where $I_{j_k}^1 = [x_{j_k-1}, x_{j_k}]$.

To keep notation simple, we label the subrectangles $R_1, \ldots, R_n$.

Now suppose that $f$ is a bdd $\mathbb{R}$-val fn. defined on $R$. Then we can define $U(f, P)$ and $L(f, P)$ as before: set $M_j = \sup_{x \in R_j} f$ and $m_j = \inf_{x \in R_j} f$ and define the upper/lower sums by $U(f, P) = \sum_{j=1}^n M_j V(R_j)$ and $L(f, P) = \sum_{j=1}^n m_j V(R_j)$.

10.1c - Def: Given partitions $P, Q$ of $R$, we say $Q$ is a refinement of $P$ if $Q(k) \subset P(k)$ for $k = 0, \ldots, d$. (Equiv, each subrect of $Q$ is cont. in a subrect of $P$.)

10.1d - Prop: If $Q$ is a refinement of $P$ and $f$ is bdd on $R$, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

10.1e - Conv: Let $P_1, P_2$ be partitions of $P$, then $L(f, P_1) \leq U(f, P_2)$.

pf: Any two partitions have a common refinement.

10.1f - Def: With $f, R$ as above define the upper and lower integrals by

$$\int_R f(x) \, dV(x) = \inf_{P} U(f, P) \text{ and } \int_R f(x) \, dV(x) = \sup_{P} L(f, P).$$

10.1g - Note: We have $\int_R f(x) \, dV(x) = \int_R f(x) \, dV(x)$

10.1h - Def: We say that $f$ is integrable on $R$ if $\int_R f(x) \, dV(x)$ exists.

In this case let $\int_R f(x) \, dV(x)$ denote this common value and call it the Riemann integral of $f$ over $R$.

10.1i - Theorem: Let $f, R$ be as above, then the following are equiv:

(i) $f$ is integrable over $R$.

(ii) $\forall \varepsilon > 0$, $\exists$ partition $P$ of $R$ st. $U(f, P) - L(f, P) < \varepsilon$.

(iii) $\exists g$ of partitions $P_n$ st. $(\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0$.

If those hold, then $\int_R f(x) \, dV(x) = \lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n)$.

Most of the results proved in 5.1 also hold in this more general context.

10.1j - Theorem: Let $f, g$ be integrable on $R$ and let $c, d \in \mathbb{R}$, then $c f + d g$ is integrable on $R$ and

$$\int_R c f(x) + d g(x) \, dV(x) = c \int_R f(x) \, dV(x) + d \int_R g(x) \, dV(x).$$

Moreover, if $f(x) \leq g(x)$ in all $x \in R$, then $\int_R f(x) \, dV(x) \leq \int_R g(x) \, dV(x)$.

10.1k - Note: Let $f$ be bdd on a rectangle $R \subset \mathbb{R}^d$ and let $P$ be a partition of $R$ with subrectangles $R_1, \ldots, R_n$. Then any sum of the form $\sum_{j=1}^n f(x_j) V(R_j)$ with $x_j \in R_j$ is called a Riemann sum; then

$$L(f, P) \leq \sum_{j=1}^n f(x_j) V(R_j) \leq U(f, P).$$