10.2 Jordan Regions

10.2a - Def: Let \( E \subset \mathbb{R}^d \). Def \( X_E : \mathbb{R}^d \to \mathbb{R} \) by \( X_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{else.} \end{cases} \)

10.2b - Notation: Let \( E \subset \mathbb{R}^d \) be bdd. Then there is a rectangle \( R \subset \mathbb{R}^d \) s.t. \( E \subset R \) and we define \( V_R(E) = \int_R X_E(x) \, dV(x) \) and \( V_R(E) = \int_R X_E(x) \, dV(x) \). Note that \( R' \subset \mathbb{R}^d \) is another rectangle with \( E \subset R' \) we have \( V_{R'}(E) = V_{R''}(E) \) and \( V_{R''}(E) = V_{R'}(E) \). So we drop \( R \) and simply write \( V(E) \) and \( V(E) \).

10.2c - Def: Say that \( E \) is a Jordan region if \( V(E) = V(E) \) and write \( V(E) = V(E) \); \( V(E) \) is called the volume of \( E \).

10.2d - Notes:
(i) A bdd set \( E \subset \mathbb{R}^d \) is a Jordan region iff \( X_E \) is int. on some vect. \( R \subset \mathbb{R}^d \) with \( E \subset R \) (and therefore any such \( R \)). In this case, \( V(E) = \int_R X_E(x) \, dV(x) \).
(ii) Every rectangle \( R \) is itself a Jordan region and its \( R = [a_1, b_1] \times \cdots \times [a_d, b_d] \), \( V(R) = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d) \) as before.
(iii) Sharp \( E, F \subset \mathbb{R}^d \) are Jordan regions with \( E \subset F \), then \( V(E) \leq V(F) \).
(iv) Let \( E \subset \mathbb{R}^d \) be bdd. If \( V(E) = 0 \), then \( E \) is a Jordan region and \( V(E) = 0 \).
(v) Sharp \( E, F \subset \mathbb{R}^d \) are bdd. If \( E \subset F \) and \( V(F) = 0 \), then \( V(E) = 0 \) and \( E \) is Jordan.
(vi) Sharp \( E, F \subset \mathbb{R}^d \) are bdd. If \( V(E) = V(F) = 0 \), then \( V(E \cup F) = 0 \) and \( E \cup F \) is Jordan.

10.2e - Def: Let \( E \subset \mathbb{R}^d \) be bdd. If \( V(E) = 0 \), then \( E \) has volume 0.

Note that by 10.1 (iv) if \( E \) has Vol. 0, then \( E \) is a Jordan region and \( V(E) = 0 \).

10.2f - Prop: Let \( E, F \subset \mathbb{R}^d \) st. \( E, F, E \cap F \) are Jordan regions. Then \( E \cap F \) is a Jordan region and \( V(E \cap F) = V(E) + V(F) - V(E \cap F) \).

10.2g - Prop: Let \( E \subset \mathbb{R}^d \) be bdd. Then \( E \) has Vol. 0 if \( \forall \varepsilon > 0 \exists \) rects. \( R_1, \ldots, R_n \)

st. \( E \subset R_1 \cup \cdots \cup R_n \) and \( \sum_i V(R_i) < \varepsilon \).

10.2h - Lemma: Let \( E \subset \mathbb{R}^d \) be bdd. Then \( V(E) = V(E) \) and \( V(E) = V(E) \).

pf: We verify (i). Since \( E \subset E \), \( V(E) \leq V(E) \) so we must show \( V(E) = V(E) \).

Then \( U(X_E, E) = \sum_i V(R_i) = V(F) \). Since \( F \) is closed \( E \subset F \) and \( V(F) = V(F) \).

Hence \( V(E) = \inf \{ u(X_E, E) : F \subset E \} \). A similar argument works to show (ii) holds.

10.2i - Theorem: If \( E \subset \mathbb{R}^d \) is Jordan, then \( E^0 \) and \( \bar{E} \) are Jordan and \( V(E^0) = V(E) = V(E) \).

pf: If \( E \) is Jordan, \( V(E) = V(E^0) \leq V(E) \) and \( V(E) = V(E) \).

10.2j - Theorem: Let \( E \subset \mathbb{R}^d \) be bdd. Then \( E \) is Jordan iff \( \partial E \) has Vol. 0.

pf: Observe that \( E \) is Jordan iff \( V(E^0) = V(E) \). Let \( R \) be a rect. st. \( E \subset R \) and let \( P \) be a partition of \( R \), then \( U(X_E, E) = U(X_E, P) - L(X_E, P) \). Sharp \( E \) is Jordan, \( \exists \) seq. of parts \( P_n \in P \) st. \( U(X_E, P_n) - L(X_E, P_n) \to 0 \) so \( U(X_E, P_n) \to 0 \).

Then \( V(\partial E) = 0 \). The converse is proved similarly.

10.2k - Cor: Sharp \( E, F \subset \mathbb{R}^d \) are Jordan regions. Then \( E \cup F, E \cap F \), and \( E \setminus F \) are all Jordan regions also.

pf: Note that the body of each of these is contained in \( 2E \).